**Harold’s Real Analysis**

**Cheat Sheet**

10 July 2025

**Number Sets**

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| **Symbol** | **Definition** | **Examples** | **Equations** | **Solution** |
| ∅ | **empty** set,  set with no members | { } |  | null |
| ℙ | **prime** numbers | {2, 3, 5, 7, 11, 13, ...} | unofficial | NA |
| ℕ | **natural** numbers | ℕ1 = {1, 2, 3, …} | Pre-2010 | NA |
| ℕ0 = {0, 1, 2, 3, …} | See ISO 80000-2 2-6.1 | |
| ℤ | **integers** | {…, −2, −1, 0, 1, 2, …} |  |  |
| ℚ | **rational** numbers | {0, ¼, ½, ¾, 1} |  |  |
| 𝔸 | **algebraic** numbers | {5, -7, ½, } |  | x is algebraic |
| 𝕋 | **transcendental** numbers | {π, e, eπ, sin(x), logb a} | 𝕋 = 𝕌 − 𝔸 | NA |
| ℝ | **real** numbers | {3.1415, -1, ⅞, } |  |  |
| 𝕀 | **imaginary** numbers | {2i, } |  |  |
| ℂ | **complex** numbers | {1 + 2i, -3.4i, ⅝} |  |  |
| 𝕌 | **universal** set | {all possible values} | ∞ | NA |
| Number set diagram | | | | |

**Derived Number Sets**

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| **Symbol** | **Definition** | **Equations** | **Examples** |
| **Integers** ℤ | | | |
| {0} | **zero** | n = 0 | {0} |
| ℤ\*  ℤ - {0}  ℤ \ {0} | **non-zero integers** | n ≠ 0 | {-3, -2, -1, 1, 2, 3, …} |
| ℤ+ | **positive integers** | n > 0 | {1, 2, 3, …} |
| ℕ ⋃ {0} | **non-negative integers** | n ≥ 0 | {0, 1, 2, 3, …} |
| ℤ‒ | **negative integers** | n < 0 | {…, -3, -2, -1} |
| ℤ‒ ⋃ {0} | **non-positive integers** | n ≤ 0 | {…, -3, -2, -1, 0} |
| **Real Numbers** ℝ | | | |
| {0} | **zero** | x = 0 | {0.0} |
| ℝ\*  ℝ - {0}  ℝ \ {0} | **non-zero real** numbers | x ≠ 0 | {-0.001, 0.001} |
| ℝ+  (0, ∞) | **positive real** numbers | x > 0 | {0.0001, 0.0002, ...} |
| ℝ+ ⋃ {0}  [0, ∞) | **non-negative real** numbers | x ≥ 0 | {0, 0.0001, 0.0002, ...} |
| ℝ‒  (-∞, 0) | **negative real** numbers | x < 0 | {…, -0.0002, -0.0001} |
| ℝ‒ ⋃ {0}  (-∞, 0] | **non-positive real** numbers | x ≤ 0 | {…, -0.0002, -0.0001, 0} |
| Z Mathematics Real Numbers - Mathematics Info | | | |

**Definitions**

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| **Term** | **Definition** |
| **Definition** | A precise and unambiguous description of the meaning of a mathematical term. It characterizes the meaning of a word by giving all the properties and only those properties that must be true. |
| **Theorem** | A mathematical statement that is proved using rigorous mathematical reasoning. In a mathematical paper, the term theorem is often reserved for the most important results. |
| **Lemma** | A minor result whose sole purpose is to help in proving a theorem. It is a steppingstone on the path to proving a theorem.  Very occasionally lemmas can take on a life of their own:  ([Zorn’s lemma](http://en.wikipedia.org/wiki/Zorn%27s_lemma), [Urysohn’s lemma](http://en.wikipedia.org/wiki/Urysohn%27s_lemma), [Burnside’s lemma](http://en.wikipedia.org/wiki/Burnside's_lemma), [Sperner’s lemma](http://en.wikipedia.org/wiki/Sperner%27s_lemma)) |
| **Corollary** | A result in which the (usually short) proof relies heavily on a given theorem (we often say that “this is a corollary of Theorem A”). |
| **Proposition** | A proved and often interesting result, but generally less important than a theorem. |
| **Conjecture** | A statement that is unproved, but is believed to be true.  ([Collatz conjectur](http://en.wikipedia.org/wiki/Collatz_conjecture)e, [Goldbach conjecture](http://en.wikipedia.org/wiki/Goldbach's_conjecture), [twin prime conjecture](http://en.wikipedia.org/wiki/Twin_prime_conjecture)) |
| **Claim** | An assertion that is then proved. It is often used like an informal lemma. |
| **Axiom** / **Postulate** | A statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proved:  ([Euclid’s five postulates](http://en.wikipedia.org/wiki/Euclidean_geometry), [Zermelo-Fraenkel axioms](http://en.wikipedia.org/wiki/Zermelo-Frankel_axioms), [Peano’s postulate](http://en.wikipedia.org/wiki/Peano_axioms)s) |
| **Identity** | A mathematical expression giving the equality of two (often variable) quantities: ([trigonometric identities](http://en.wikipedia.org/wiki/List_of_trigonometric_identities), [Euler’s identity](http://en.wikipedia.org/wiki/Euler%27s_identity)) |
| **Paradox** | A statement that can be shown, using a given set of axioms and definitions, to be both true and false. Paradoxes are often used to show the inconsistencies in a flawed theory (Russell’s paradox). The term paradox is often used informally to describe a surprising or counterintuitive result that follows from a given set of rules ([Banach-Tarski paradox](http://en.wikipedia.org/wiki/Banach%E2%80%93Tarski_paradox), [Alabama paradox](http://en.wikipedia.org/wiki/Alabama_paradox#Alabama_Paradox), [Gabriel’s horn](http://en.wikipedia.org/wiki/Gabriel%27s_Horn)). |

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| Corollary  ↑  **Theorem** / Proposition  ↑  Lemma  ↑  Axiom / Postulate  ↑  Conjecture / Claim  ↑  Definition | Venn Diagram of Sets Used to Construct the Real Numbers |

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| **Textbook** | Bloch, Ethan D., The Real Numbers and Real Analysis. Springer New York, 2011. |

**Ch. 1.2: Natural Numbers** ℕ

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| **Axiom / Theorem / Lemma / Definition** | **Description** |
| **Operations: Binary, Unary**  **(Definition 1.1.1)** | Let S be a set.  A **binary** operation on S is a function S × S → S.  A **unary** operation on S is a function S → S. |
| **Peano Postulates**  **(Axiom 1.2.1)** | There exists a set ℕ with an element 1 ∈ ℕ and a function s: ℕ → ℕ that satisfy the following three properties.  a. There is no n ∈ ℕ such that s(n) = 1.  b. The function s is injective.  c. Let G ⊆ ℕ be a set. Suppose that 1 ∈ G, and that if g ∈ G then s(g) ∈ G. Then G = ℕ. |
| **Natural Number**  **(Definition 1.2.2)** | The set of **natural numbers**, denoted ℕ, is the set the existence of which is given in the Peano Postulates. |
| **Lemma 1.2.3** | Let a ∈ ℕ. Suppose that a ≠ 1.  Then there is a unique b ∈ ℕ such that a = s(b). |
| **Definition by Recursion**  **(Theorem 1.2.4)** | Let H be a set, let e ∈ H and let k: H → H be a function. Then there is a unique function f: ℕ → H such that f(1) = e, and that f ◦ s = k ◦ f. |
| **Operation: +**  **(Theorem 1.2.5)** | There is a unique binary operation +: ℕ × ℕ → ℕ that satisfies the following two properties for all n,m ∈ ℕ.  a. n + 1 = s(n). (successor).  b. n + s(m) = s(n + m). [= n + (m+1)] |
| **Operation: \***  **(Theorem 1.2.6)** | There is a unique binary operation \*: ℕ × ℕ → ℕ that satisfies the following two properties for all n,m ∈ ℕ.  a. n \* 1 = n.  b. n \* s(m) = n(m+1) = (n \* m) + n. |
| **Addition Laws**  **(Theorem 1.2.7a)** | Let a, b, c ∈ ℕ.  1. If a + c = b + c, then a = b (Cancellation Law for Addition).  2. (a + b) + c = a + (b + c) (Associative Law for Addition).  3. 1 + a = s(a) = a + 1.  4. a + b = b + a (Commutative Law for Addition).  5. a + b ≠ 1.  6. a + b ≠ a. |
| **Multiplication Laws**  **(Theorem 1.2.7b)** | Let a, b, c ∈ ℕ.  7. a \* 1 = a = 1 \* a (Identity Law for Multiplication).  8. (a + b)c = ac + bc (Distributive Law).  9. ab = ba (Commutative Law for Multiplication).  10. c(a + b) = ca + cb (Distributive Law).  11. (ab)c = a(bc) (Associative Law for Multiplication).  12. If ac = bc then a = b (Cancellation Law for Multiplication).  13. ab = 1 if and only if a = 1 = b. |
| **Relation: <**  **(Definition 1.2.8a)** | The relation < on ℕ is defined by a < b if and only if there is some p ∈ N such that a + p = b, for all a,b ∈ N. |
| **Relation: ≤**  **(Definition 1.2.8b)** | The relation ≤ on ℕ is defined by a ≤ b if and only if a < b or a = b, for all a,b ∈ ℕ. |
| **Relation: < and ≤**  **(Theorem 1.2.9)** | Let a, b, c, d ∈ ℕ.  1. a ≤ a, and a ≮ a, and a < a + 1.  2. 1 ≤ a.  3. If a < b and b < c, then a < c; if a ≤ b and b < c, then a < c; if a < b and b ≤ c, then a < c; if a ≤ b and b ≤ c, then a ≤ c.  4. a < b if and only if a + c < b + c.  5. a < b if and only if ac < bc.  6. Precisely one of a < b or a = b or a > b holds (Trichotomy Law).  7. a ≤ b or b ≤ a.  8. If a ≤ b and b ≤ a, then a = b.  9. It cannot be that b < a < b + 1.  10. a ≤ b if and only if a < b + 1.  11. a < b if and only if a + 1 ≤ b. |
| **Well-Ordering Principle**  **(Theorem 1.2.10)** | Let G ⊆ ℕ be a non-empty set. Then there is some m ∈ G such that m ≤ g for all g ∈ G. |

**Ch. 1.3 – 1.4: Integers** ℤ

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| **Axiom, Theorem, etc.** | **Description** |
| **Relation: ~**  **(Definition 1.3.1)** | The relation ∼ on ℕ × ℕ is defined by (a,b) ∼ (c,d) if and only if a + d = b + c, for all (a,b),(c,d) ∈ ℕ × ℕ. |
| **Relation: ~**  **(Lemma 1.3.2)** | The relation ∼ is an equivalence relation on ℕ × ℕ. |
| **Integers:** ℤ  **(Definition 1.3.3)** | The set of integers, denoted ℤ, is the set of equivalence classes of ℕ × ℕ with respect to the equivalence relation ∼. |
| **Well-Defined: +, \***  **(Lemma 1.3.4)** | The binary operations + and \*, the unary operation −, and the relation <, all on ℤ, are well-defined. |
| **Addition & Multiplication Laws**  **(Definition 1.4.1 & 1.3.5)** | An **ordered integral domain** is a set R with elements 0,1 ∈ R, binary operations + and ·, a unary operation − and a relation <, which satisfy the following properties.  Let x, y, z ∈ R.  a. (x + y) + z = x + (y + z) (Associative Law for Addition).  b. x + y = y + x (Commutative Law for Addition).  c. x + 0 = x (Identity Law for Addition).  d. x + (−x) = 0 (Inverses Law for Addition).  e. (xy)z = x(yz) (Associative Law for Multiplication).  f. xy = yx (Commutative Law for Multiplication).  g. x · 1 = x (Identity Law for Multiplication).  h. x(y + z) = xy + xz (Distributive Law).  i. If xy = 0, then x = 0 or y = 0 (No Zero Divisors Law).  j. Precisely one of x < y or x = y or x > y holds (Trichotomy Law).  k. If x < y and y < z, then x < z (Transitive Law).  l. If x < y then x + z < y + z (Addition Law for Order).  m. If x < y and z > 0, then xz < yz (Multiplication Law for Order).  n. 0 ≠ 1 (Non-Triviality). |
| **Relation: ≤**  **(Definition 1.4.2)** | Let R be an ordered integral domain, and let A ⊆ R be a set.  1. The relation ≤ on R is defined by a ≤ b if and only if a < b or a = b, for all a,b ∈ R.  2. The set A has a least element if there is some a ∈ A such that a ≤ x for all x ∈ A. |
| **Well-Ordering Principle**  **(Definition 1.4.3)** | Let R be an ordered integral domain. The ordered integral domain R satisfies the **Well-Ordering Principle** if every non-empty subset of {x ∈ R | x > 0} has a least element. |
| **Axiom for the Integers**  **(Axiom 1.4.4)** | There exists an ordered integral domain ℤ that satisfies the Well-Ordering Principle. |
| **Properties of Integers**  **(Lemma 1.4.5 & 1.3.8)** | Let x, y, z ∈ ℤ.  1. If x + z = y + z, then x = y (Cancellation Law for Addition).  2. −(−x) = x.  3. −(x + y) = (−x) + (−y).  4. x · 0 = 0.  5. If z ≠ 0 and if xz = yz, then x = y (Cancellation Law for Mult.).  6. (−x)y = −xy = x(−y).  7. xy = 1 if and only if x = 1 = y or x = −1 = y.  8. x > 0 if and only if −x < 0, and x < 0 if and only if −x > 0.  9. 0 < 1.  10. If x ≤ y and y ≤ x, then x = y.  11. If x > 0 and y > 0, then xy > 0. If x > 0 and y < 0, then xy < 0. |
| **Discreteness**  **(Theorem 1.4.6 & 1.3.9)** | Let x ∈ ℤ. Then there is no y ∈ ℤ such that x < y < x + 1. |
| **Positive/Negative: +, -**  **(Definition 1.4.7 & 1.3.6)** | 1. Let x ∈ ℤ. The number x is positive if x > 0, and the number x is negative if x < 0. |
| ℕ ⊆ ℤ:  **(Theorem 1.3.7 & Definition 1.4.7)** | Let i: ℕ →ℤ be defined by i(n) = [(n+1,1)] for all n ∈ ℕ.  1. The function i: ℕ → ℤ is injective.  2. i(ℕ) = {x ∈ ℤ | x > 0ˆ}.  3. i(1) = 1ˆ.  4. Let a,b ∈ ℕ. Then  a. i(a+b) = i(a) + i(b);  b. i(ab) = i(a) i(b);  c. a < b if and only if i(a) < i(b). |
| **Natural Numbers:** ℕ  **(Definition 1.4.7)** | 2. The set of natural numbers, denoted ℕ, is defined by ℕ = {x ∈ ℤ | x > 0}. |
| **Peano Postulates**  **(Theorem 1.4.8 & Axiom 1.2.1)** | Let s: ℕ → ℕ be defined by s(n) = n + 1 for all n ∈ ℕ.  a. There is no n ∈ ℕ such that s(n) = 1.  b. The function s is injective.  c. Let G ⊆ ℕ be a set. Suppose that 1 ∈ G, and that if g ∈ G then s(g) ∈ G. Then G = ℕ. |

**Ch. 1.5: Rational Numbers** ℚ

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| **Definition / Lemma / Theorem** | **Description** |
| **Relation:** ≍,ℤ∗  **(Definition 1.5.1)** | Let ℤ∗ = ℤ− {0}. The relation ≍ on ℤ × ℤ∗ is defined by (x, y) ≍ (z,w) if and only if xw = yz, for all (x, y),(z,w) ∈ ℤ × ℤ∗. |
| **Relation:** ≍  **(Lemma 1.5.2)** | The relation ≍ is an equivalence relation. |
| **Rational Numbers:** ℚ  **(Definition 1.5.3)** | The set of rational numbers, denoted ℚ, is the set of equivalence classes of ℤ × ℤ∗ with respect to the equivalence relation ≍.  The elements 0¯,1¯ ∈ ℚ are defined by 0¯ = [(0,1)] and 1¯ = [(1,1)]. Let ℚ∗ = ℚ − {0¯}. The binary operations + and · on ℚ are defined by  [(x,y)] + [(z,w)] = [(xw + yz,yw)]  [(x,y)] · [(z,w)] = [(xz, yw)]  for all [(x, y)],[(z,w)] ∈ ℚ.   * −: The unary operation − on ℚ is defined by −[(x, y)] = [(−x, y)] for all [(x,y)] ∈ ℚ. * −1: The unary operation −1 on ℚ∗ is defined by [(x, y)]−1 = [(y, x)] for all [(x, y)] ∈ ℚ∗. * <: The relation < on ℚ is defined by [(x,y)] < [(z,w)] if and only if either xw < yz when y > 0 and w > 0 or when y < 0 and w < 0, * >: The relation > on ℚ is defined by [(x,y)] > [(z,w)] if and only if either xw > yz when y > 0 and w < 0 or when y < 0 and w > 0, for all [(x, y)],[(z,w)] ∈ ℚ. * ≤: The relation ≤ on ℚ is defined by [(x, y)] ≤ [(z,w)] if and only if [(x,y)] < [(z,w)] or [(x,y)] = [(z,w)], for all [(x,y)],[(z,w)] ∈ ℚ. |
| **Well-Defined**: ℚ  **(Lemma 1.5.4)** | The binary operations + and ·, the unary operations − and −1, and the relation <, all on ℚ, are **well-defined**. |
| **Addition and Multiplication Laws**  **(Theorem 1.5.5)** | Let r,s,t ∈ ℚ.  **Field:**  1. (r + s) + t = r + (s + t) (Associative Law for Addition).  2. r + s = s + r (Commutative Law for Addition).  3. r + 0¯ = r (Identity Law for Addition).  4. r + (−r) = 0¯ (Inverses Law for Addition).  5. (rs)t = r(st) (Associative Law for Multiplication).  6. rs = sr (Commutative Law for Multiplication).  7. r· 1¯ = r (Identity Law for Multiplication).  8. If r ≠ 0¯, then r · r−1 = 1¯ (Inverses Law for Multiplication).  9. r(s + t) = rs + rt (Distributive Law).  **Ordered Field:**  11. If r < s and s < t, then r < t (Transitive Law).  12. If r < s then r + t < s + t (Addition Law for Order).  13. If r < s and t > 0¯, then rt < st (Multiplication Law for Order).  14. 0¯ ≠ 1¯ (Non-Triviality). |
| ℤ ⊆ ℚ:  **(Theorem 1.5.6)** | Let i: ℤ → ℚ be defined by i(x) = [(x,1)] for all x ∈ ℤ.  1. The function i: ℤ → ℚ is injective.  2. i(0) = 0¯ and i(1) = 1¯.  3. Let x, y ∈ ℤ. Then  a. i(x + y) = i(x) + i(y);  b. i(−x) = −i(x);  c. i(xy) = i(x) i(y);  d. x < y if and only if i(x) < i(y).  4. For each r ∈ ℚ there are x,y ∈ ℤ such that y ≠ 0 and r = i(x) (i(y))−1. |
| **Operations: -**,÷, s−1,  **(Definition 1.5.7)** | The binary operation − on ℚ is defined by r − s = r + (−s) for all r,s ∈ ℚ.  The binary operation ÷ on ℚ∗ is defined by r ÷ s = rs−1 for all r,s ∈ ℚ∗;  we also let 0 ÷ s = 0 · s−1 = 0 for all s ∈ ℚ∗.  The number r ÷ s is also denoted . |
| **Rational Numbers:** ℚ  **(Lemma 1.5.8)**  **(Definition 1.5.3 Restated)** | Let a,c ∈ ℤ and b,d ∈ ℤ∗.  1. = if and only if ad = bc.  2. + = .  3. − = .  4. · = .  5. If a ≠ 0, then = .  6. If b > 0 and d > 0, or if b < 0 and d < 0, then < if and only if ad < bc;  if b > 0 and d < 0, or if b < 0 and d > 0, then > if and only if ad > bc. |

**Ch. 1.6: Dedekind Cuts Dr**

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| **Definition / Lemma** | **Description** |
| **Dedekind cut**  **(Definition 1.6.1)**  **AKA “upper cut”** | Let A ⊆ ℚ be a set. The set A is a **Dedekind cut** if the following three properties hold.  a. A ≠ 0 and A ≠ ℚ.  b. Let x ∈ A. If y ∈ ℚ and y ≥ x, then y ∈ A.  c. Let x ∈ A. Then there is some y ∈ A such that y < x. |
| **Interpreting Dedekind cuts** | A Dedekind cut is a set, A, of rational numbers, with the  properties shown above.    a. Property (a) says A must be nonempty and cannot be all of ℚ.  b. Property (b) says if a number, x, is in A, then all rational numbers greater than x are also in A.  c. Property (c) is where things get interesting. It says that if x is in A, then there is at least one element of A that is smaller than x. (Actually, there are infinitely many.) This property is what is going to allow us to fill in the gaps in the rational numbers. |
| **Dedekind cut Existence**  **(Lemma 1.6.2)** | Let r ∈ ℚ. Then the set {x ∈ ℚ | x > r} is a Dedekind cut. |
| **Dedekind cut not in form of Lemma 1.6.2**  **(Example 1.6.3)** | Let  T = {x ∈ ℚ | x > 0 and x2 > 2}. (1.6.1)  It is seen by Exercise 1.6.2 (1) that T is a Dedekind cut, and by Part (2) of that exercise it is seen that if T has the form {x ∈ ℚ | x > r} for some r ∈ ℚ, then r2 = 2. By **Theorem 2.6.11** we know that there is no rational number x such that x2 = 2, and it follows that T is a Dedekind cut that is not of the form given in Lemma 1.6.2. |
| **Rational cut Dr**  **(Definition 1.6.4)** | Let r ∈ ℚ.  The **rational cut** at r, denoted Dr, is the Dedekind cut Dr = {x ∈ ℚ | x > r}.  An **irrational cut** is a Dedekind cut that is not a rational cut at any rational number. |
| **Complement of Dedekind cut**  **(Lemma 1.6.5)** | Let A ⊆ ℚ be a Dedekind cut.  1. ℚ − A = {x ∈ ℚ | x < a for all a ∈ A}. or { x ∈ ℚ | x ≤ r }.  2. Let x ∈ ℚ − A. If y ∈ ℚ and y ≤ x, then y ∈ ℚ − A. |
| **Trichotomy Law**  **(Lemma 1.6.6)** | Let A,B ⊆ ℚ be Dedekind cuts. Then precisely one of A ⫋ B or A = B or B ⫋ A holds.  **NOTE**: A ⫋ B means that both A ⊂ B and A ≠ B. |
| **Union of Family of Sets**  **(Lemma 1.6.7)** | Let A be a non-empty family of subsets of ℚ. Suppose that X is a Dedekind cut for all X ∈ A. If X ≠ ℚ, then X is a Dedekind cut.  For example, think about what happens if the set A is defined this way:  A = {x ∈ ℚ | x > 4},  {x ∈ ℚ | x > 3.2},  {x ∈ ℚ | x > 3.15},  {x ∈ ℚ | x > 3.142},  {x ∈ ℚ | x > 3.1416},  {x ∈ ℚ | x > 3.14160},  {x ∈ ℚ | x > 3.141593}, …}  If you were to union all of the elements of A, you would end up with {x ∈ ℚ | x > π}. This is how the “gaps” get filled in. |
| **Dedekind cut Examples**  **(Lemma 1.6.8)** | Let A,B ⊆ ℚ be Dedekind cuts.  1. The set {r ∈ ℚ | r = a + b for some a ∈ A and b ∈ B} is a Dedekind cut.  2. The set {r ∈ ℚ | −r < c for some c ∈ ℚ − A} is a Dedekind cut.  3. Suppose that 0 ∈ ℚ − A and 0 ∈ ℚ − B. The set {r ∈ ℚ | r = ab for some a ∈ A and b ∈ B} is a Dedekind cut.  4. Suppose that there is some q ∈ ℚ − A such that q > 0. The set {r ∈ ℚ | r > 0 and < c for some c ∈ ℚ − A} is a Dedekind cut. |
| **Well-Ordering Principle**  **(Lemma 1.6.9)** | Let A ⊆ ℚ be a Dedekind cut. Let y ∈ ℚ.  1. Suppose that y > 0. Then there are u ∈ A and v ∈ ℚ − A such that y = u − v, and v < e for some e ∈ ℚ − A.  2. Suppose that y > 1, and that there is some q ∈ ℚ − A such that q > 0. Then there are r ∈ A and s ∈ ℚ − A such that s > 0, and y > , and s < g for some g ∈ ℚ − A. |

**Ch. 1.7: Real Numbers** ℝ **(Ch. 1)**

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| **Axiom / Theorem / Lemma / Definition** | **Description** |
| **Real Numbers:** ℝ  **Definition 1.7.1** | The set of real numbers, denoted ℝ, is defined by  ℝ = {A ⊆ ℚ | A is a Dedekind cut}. |
| **Relations: <, ≤**  **(Definition 1.7.2)** | The relation < on ℝ is defined by  A < B if and only if A ⫌ B, for all A,B ∈ ℝ.  The relation ≤ on ℝ is defined by  A ≤ B if and only if A ⊇ B, for all A,B ∈ ℝ. |
| **Operation: +, −**  **(Definition 1.7.3)** | The binary operation + on ℝ is defined by  A + B = {r ∈ ℚ | r = a + b for some a ∈ A and b ∈ B}  for all A,B ∈ ℝ.  The unary operation − on ℝ is defined by  −A = {r ∈ ℚ | −r < c for some c ∈ ℚ − A}  for all A ∈ ℝ. |
| **Multiply Operator Setup**  **Lemma 1.7.4** | Let A ∈ ℝ, and let r ∈ ℚ.  1. A > Dr if and only if there is some q ∈ ℚ − A such that q > r.  2. A ≥ Dr if and only if r ∈ ℚ − A if and only if a > r for all a ∈ A.  3. If A < D0 then −A ≥ D0. |
| **Operations:** •, -1  **(Definition 1.7.5)** | The binary operation • on ℝ is defined by  A • B =  The unary operation −1 on ℝ − { D0 } is defined by  A-1 = |
| **Addition and Multiplication Laws**  **(Theorem 1.7.6)** | Let A,B,C ∈ ℝ.  **Field:**  1. (A + B) + C = A + (B + C) (Associative Law for Addition).  2. A + B = B + A (Commutative Law for Addition).  3. A + D0 = A (Identity Law for Addition).  4. A + (−A) = D0 = 0 (Inverses Law for Addition).  5. (AB)C = A(BC) (Associative Law for Multiplication).  6. AB = BA (Commutative Law for Multiplication).  7. A • D1 = A (Identity Law for Multiplication).  8. If A ≠ D0, then AA−1 = D1 =1 (Inverses Law for Multiplication).  9. A(B + C) = AB + AC (Distributive Law).  **Ordered Field:**  10. Precisely one of A < B or A = B or A > B holds (Trichotomy Law).  11. If A < B and B < C, then A < C (Transitive Law).  12. If A < B then A + C < B + C (Addition Law for Order).  13. If A < B and C > D0, then AC < BC (Multiplication Law for Order).  14. D0 < D1 or 0 < 1 (Non-Triviality). |
| **Least Upper Bound Property Setup**  **(Definition 1.7.7)** | Let A ⊆ ℝ be a set.  1. The set A is **bounded above** if there is some M ∈ ℝ such that X ≤ M for all X ∈ A. The number M is called an upper bound of A.  2. The set A is **bounded below** if there is some P ∈ ℝ such that X ≥ P for all X ∈ A. The number P is called a lower bound of A.  3. The set A is **bounded** if it is bounded above and bounded below.  4. Let M ∈ ℝ. The number M is a **least upper bound** (also called a **supremum**) of A if M is an upper bound of A, and if M ≤ T for all upper bounds T of A.  5. Let P ∈ ℝ. The number P is a **greatest lower bound** (also called an **infimum**) of A if P is a lower bound of A, and if P ≥ V for all lower bounds V of A. |
| **Greatest Lower Bound Property (glb)**  **(Theorem 1.7.8)** | Let A ⊆ ℝ be a set. If A is non-empty and bounded below, then A has a greatest lower bound. (used in Dedekind cut proofs) |
| **Least Upper Bound Property (lub)**  **(Theorem 1.7.9)** | Let A ⊆ ℝ be a set. If A is nonempty and bounded above, then A has a least upper bound. |
| ℚ ⊆ ℝ:  **(Theorem 1.7.10)** | Let i: ℚ → ℝ be defined by i(r) = Dr for all r ∈ ℝ.  1. The function i: ℚ → ℝ is injective.  2. i(0) = D0 and i(1) = D1.  3. Let r,s ∈ ℚ. Then  a. i(r +s) = i(r) +i(s);  b. i(−r) = −i(r);  c. i(rs) = i(r) i(s);  d. if r ≠ 0 then i(r−1) = [i(r)]−1;  e. r < s if and only if i(r) < i(s). |

**Ch. 2.2: Real Numbers** ℝ

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| **Definitions / Axiom** | **Description** |
| **Addition and Multiplication Laws**  **(Definition 2.2.1)** | An ordered field is a set F with elements 0,1 ∈ F, binary operations + and ·, a unary operation −, a relation <, and a unary operation −1 on F − {0}, which satisfy the following properties.  Let x,y,z ∈ F.  a. (x + y) + z = x + (y + z) (Associative Law for Addition).  b. x + y = y + x (Commutative Law for Addition).  c. x + 0 = x (Identity Law for Addition).  d. x + (−x) = 0 (Inverses Law for Addition).  e. (xy)z = x(yz) (Associative Law for Multiplication).  f. xy = yx (Commutative Law for Multiplication).  g. x · 1 = x (Identity Law for Multiplication).  h. If x ≠ 0, then xx−1 = 1 (Inverses Law for Multiplication).  i. x(y + z) = xy + xz (Distributive Law).  j. Precisely one of x < y or x = y or x > y holds (Trichotomy Law).  k. If x < y and y < z, then x < z (Transitive Law).  l. If x < y then x + z < y + z (Addition Law for Order).  m. If x < y and z > 0, then xz < yz (Multiplication Law for Order).  n. 0 ≠ 1 (Non-Triviality). |
| **Bounds**  **(Definition 2.2.2)** | Let F be an ordered field and let A ⊆ F be a set.  1. The set A is **bounded above** if there is some M ∈ F such that x ≤ M for all x ∈ A. The number M is called an **upper bound** of A.  2. The set A is **bounded below** if there is some P ∈ F such that x ≥ P for all x ∈ A. The number P is called a **lower bound** of A.  3. The set A is **bounded** if it is bounded above and bounded below.  4. Let M ∈ F. The number M is a **least upper bound** (also called a **supremum**) of A if M is an upper bound of A, and if M ≤ T for all upper bounds T of A.  5. Let P ∈ F. The number P is a **greatest lower bound** (also called an **infimum**) of A if P is a lower bound of A, and if P ≥ V for all lower bounds V of A. |
| **Least Upper Bound Property**  **(Definition 2.2.3)** | Let F be an ordered field. The ordered field F satisfies the **Least Upper Bound Property** if every non-empty subset of F that is bounded above has a least upper bound. |
| **Axiom for the Real Numbers**  **(Axiom 2.2.4)** | There exists an ordered field ℝ that satisfies the Least Upper Bound Property. |

**Ch. 2.3: Algebraic Properties of Real Numbers** ℝ

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| **Definitions / Axiom** | **Description** |
| **Operators: −, ÷, 2, ≤, 2**  **(Definition 2.3.1)** | 1a. The binary operation − on ℝ is defined by a − b = a + (−b) for all a,b ∈ ℝ.  1b. The binary operation ÷ on ℝ − {0} is defined by a ÷ b = ab−1 for all a,b ∈ ℝ − {0}; we also let 0 ÷ s = 0 ·s−1 = 0 for all s ∈ ℝ − {0}. The number a ÷ b is also denoted or a/b.  2. Let a ∈ ℝ. The **square** of a, denoted a2, is defined by a2 = a · a.  3. The relation ≤ on ℝ is defined by x ≤ y if and only if x < y or x = y, for all x,y ∈ ℝ.  4. The number 2 ∈ ℝ is defined by 2 = 1 + 1. |
| **Properties of Real Numbers**  **(Lemma 2.3.2)** | Let a,b,c ∈ ℝ.  1. If a + c = b + c then a = b (Cancellation Law for Addition).  2. If a + b = a then b = 0.  3. If a + b = 0 then b = −a.  4. −(a + b) = (−a) + (−b).  5. −0 = 0.  6. If ac = bc and c ≠ 0, then a = b (Cancellation Law for Multiplication).  7. 0 · a = 0 = a · 0.  8. If ab = a and a ≠ 0, then b = 1.  9. If ab = 1 then b = a−1.  10. If a ≠ 0 and b ≠ 0, then (ab)−1 = a−1 b−1.  11. (−1)· a = −a.  12. (−a)b = −ab = a(−b).  13. −(−a) = a.  14. (−1)2 = 1 and 1−1 = 1.  15. If ab = 0, then a = 0 or b = 0 (No Zero Divisors Law).  16. If a ≠ 0 then (a−1)−1 = a.  17. If a ≠ 0 then (−a)−1 = −a−1. |
| **Relations: <, ≤**  **(Lemma 2.3.3)** | Let a,b,c,d ∈ ℝ.  1. If a ≤ b and b ≤ a, then a = b.  2. If a ≤ b and b ≤ c, then a ≤ c.  If a ≤ b and b < c, then a < c.  If a < b and b ≤ c, then a < c.  3. If a ≤ b then a + c ≤ b + c.  4. If a < b and c < d, then a + c < b + d;  if a ≤ b and c ≤ d, then a + c ≤ b + d.  5. a > 0 if and only if −a < 0, and a < 0 if and only if −a > 0; also  a ≥ 0 if and only if −a ≤ 0, and a ≤ 0 if and only if −a ≥ 0.  6. a < b if and only if b − a > 0 if and only if −b < −a; also  a ≤ b if and only if b−a ≥ 0 if and only if −b ≤ −a.  7. If a ≠ 0 then a2 > 0.  8. −1 < 0 < 1.  9. a < a + 1.  10. If a ≤ b and c > 0, then ac ≤ bc.  11. If 0 ≤ a < b and 0 ≤ c < d, then ac < bd;  if 0 ≤ a ≤ b and 0 ≤ c ≤ d, then ac ≤ bd.  12. If a < b and c < 0, then ac > bc.  13. If a > 0 then a−1 > 0.  14. If a > 0 and b > 0, then a < b if and only if b−1 < a−1 if and only if a2 < b2. |
| **Positive / Negative**  **(Definition 2.3.4)** | Let a ∈ ℝ.  The number a is **positive** if a > 0;  the number a is **negative** if a < 0; and  the number a is **non-negative** if a ≥ 0. |
| **Positive / Negative**  **(Lemma 2.3.5)** | Let a,b,c,d ∈ ℝ.  1. If a > 0 and b > 0, then a + b > 0. (Addition)  If a > 0 and b ≥ 0, then a + b > 0.  If a ≥ 0 and b ≥ 0, then a + b ≥ 0.  2. If a < 0 and b < 0, then a + b < 0.  If a < 0 and b ≤ 0, then a + b < 0.  If a ≤ 0 and b ≤ 0, then a + b ≤ 0.  3. If a > 0 and b > 0, then ab > 0. (Multiplication)  If a > 0 and b ≥ 0, then ab ≥ 0.  If a ≥ 0 and b ≥ 0, then ab ≥ 0.  4. If a < 0 and b < 0, then ab > 0.  If a < 0 and b ≤ 0, then ab ≥ 0.  If a ≤ 0 and b ≤ 0, then ab ≥ 0.  5. If a < 0 and b > 0, then ab < 0.  If a < 0 and b ≥ 0, then ab ≤ 0.  If a ≤ 0 and b > 0, then ab ≤ 0.  If a ≤ 0 and b ≥ 0, then ab ≤ 0. |
| **Intervals**  **(Definition 2.3.6)** | Let a,b ∈ ℝ.  An **open bounded interval** is a set of the form  (a,b) = {x ∈ ℝ | a < x < b}, where a ≤ b.  A **closed bounded interval** is a set of the form  [a,b] = {x ∈ ℝ | a ≤ x ≤ b}, where a ≤ b.  A **half-open interval** is a set of the form  [a,b) = {x ∈ ℝ | a ≤ x < b} or (a,b] = {x ∈ ℝ | a < x ≤ b}, where a ≤ b.  An **open unbounded interval** is a set of the form  (a,∞) = {x ∈ ℝ | a < x} or (−∞,b) = {x ∈ ℝ | x < b} or (−∞,∞) = ℝ.  A **closed unbounded interval** is a set of the form  [a,∞) = {x ∈ ℝ | a ≤ x} or (−∞,b] = {x ∈ ℝ | x ≤ b}. |
| **Interval Types** | * An **open interval** is either an open bounded interval or an open unbounded interval. * A **closed interval** is either a closed bounded interval or a closed unbounded interval. * A **right unbounded interval** is any interval of the form (a,∞), [a,∞) or (−∞,∞). * A **left unbounded interval** is any interval of the form (−∞,b), (−∞,b] or (−∞,∞). * A **non-degenerate interval** is any interval of the form (a,b), (a,b], [a,b) or [a,b] where a < b, or any unbounded interval. * The number a in intervals of the form [a,b), [a,b] or [a, ∞) is called the **left endpoint** of the interval. * The number b in intervals of the form (a,b], [a,b] or (−∞,b] is called the **right endpoint** of the interval. * An **endpoint** of an interval is either a left endpoint or a right endpoint. * The **interior** of an interval is everything in the interval other than its endpoints. |
| **Intervals**  **(Lemma 2.3.7)** | Let I ⊆ ℝ be an interval.  1. If x, y ∈ I and x ≤ y, then [x, y] ⊆ I.  2. If I is an open interval, and if x ∈ I, then there is some δ > 0 such that [x − δ, x + δ] ⊆ I. |
| **Absolute Value**  **(Definition 2.3.8)** | Let a ∈ ℝ. The absolute value of a, denoted |a|, is defined by  |a| = (a, if a ≥ 0 −a, if a < 0. |
| **Properties of Absolute Value**  **(Lemma 2.3.9)** | Let a,b ∈ ℝ.  1. |a| ≥ 0, and |a| = 0 if and only if a = 0.  2. −|a| ≤ a ≤ |a|.  3. |a| = |b| if and only if a = b or a = −b.  4. |a| < b if and only if −b < a < b, and |a| ≤ b if and only if −b ≤ a ≤ b.  5. |ab| = |a|·|b|.  6. |a + b| ≤ |a| + |b| (Triangle Inequality).  7. ||a| − |b|| ≤ |a + b| and ||a| − |b|| ≤ |a − b|. |
| **Epsilon: ε ≈ 0**  **(Lemma 2.3.10)** | Let a ∈ ℝ.  1. a ≤ 0 if and only if a < ε for all ε > 0.  2. a ≥ 0 if and only if a > −ε for all ε > 0.  3. a = 0 if and only if |a| < ε for all ε > 0. |

**2.4 Real Numbers Include Natural, Integers, and Rationals (**ℕ ⊂ ℤ ⊂ ℚ ⊂ ℝ**)**

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| **Theorem / Lemma / Definition / Corollary** | **Description** |
| **Inductive Set**  **(Definition 2.4.1)** | Let S ⊆ ℝ be a set. The set S is **inductive** if it satisfies the following two properties.  (a) 1 ∈ S.  (b) If a ∈ S, then a + 1 ∈ S. |
| **Definition:** ℕ  **(Definition 2.4.2)** | The set of **natural numbers**, denoted ℕ, is the intersection of all inductive subsets of ℝ. |
| **Properties of** ℕ  **(Lemma 2.4.3)** | 1. ℕ is inductive.  2. If A ⊆ ℝ and A is inductive, then ℕ ⊆ A.  3. If n ∈ ℕ then n ≥ 1. |
| **Peano Postulates**  **(Theorem 2.4.4)** | Let s: N → N be defined by s(n) = n + 1 for all n ∈ ℕ.  a. There is no n ∈ ℕ such that s(n) = 1.  b. The function s is injective.  c. Let G ⊆ ℕ be a set. Suppose that 1 ∈ G, and that if g ∈ G then s(g) ∈ G. Then G = ℕ. |
| ℕ **Closed Under +,** ·  **(Lemma 2.4.5)** | Let a,b ∈ ℕ. Then a + b ∈ ℕ and ab ∈ ℕ. |
| **Well-Ordering Principle**  **(Theorem 2.4.6)** | Let G ⊆ ℕ be a non-empty set. Then there is some m ∈ G such that m ≤ g for all g ∈ G. |
| **Definition:** ℤ  **(Definition 2.4.7)** | Let − ℕ = {x ∈ ℝ | x = −n for some n ∈ ℕ }.  The set of **integers**, denoted ℤ, is defined by ℤ = − ℕ ⋃ {0} ⋃ ℕ. |
| **Properties of** ℤ  **(Lemma 2.4.8)** | 1. ℕ ⊆ ℤ.  2. a ∈ ℕ if and only if a ∈ ℤ and a > 0.  3. The three sets − ℕ, {0} and ℕ are mutually disjoint. |
| ℤ **Closed Under +,** ·, −  **(Lemma 2.4.9)** | Let a,b ∈ ℤ. Then a + b ∈ ℤ, and ab ∈ ℤ, and −a ∈ ℤ. |
| **Integers are Discrete**  **(Theorem 2.4.10)** | Let a,b ∈ ℤ.  1. If a < b then a + 1 ≤ b.  2. There is no c ∈ ℤ such that a < c < a + 1.  3. If |a−b| < 1 then a = b. |
| **Definition:** ℚ  **(Definition 2.4.11)** | The set of **rational numbers**, denoted ℚ, is defined by  ℚ = {x ∈ ℝ | x = a / b for some a,b ∈ ℤ such that b ≠ 0}.  The set of **irrational numbers** is the set ℝ − ℚ. |
| **Properties of** ℚ  **(Lemma 2.4.12)** | 1. ℤ ⊆ ℚ.  2. q ∈ ℚ and q > 0 if and only if q = a / b for some a,b ∈ ℕ. |
| **Fraction Manipulation**  **(Lemma 2.4.13)** | Let a,b,c,d ∈ ℤ. Suppose that b ≠ 0 and d ≠ 0.  1. a / b = 0 if and only if a = 0.  2. a / b = 1 if and only if a = b.  3. a / b = c / d if and only if ad = bc.  4. a / b + c / d = (ad + bc) / bd.  5. –(a / b) = (−a) / b = a / (−b).  6. a / b · c / d = ac / bd.  7. If a ≠ 0, then (a / b)−1 = b / a. |
| ℚ **Closed Under +,** ·, −, −1  **(Corollary 2.4.14)** | Let a,b ∈ ℚ. Then a + b ∈ ℚ, and ab ∈ ℚ, and −a ∈ ℚ, and if a ≠ 0 then a−1 ∈ ℚ. |

Diagram

Description automatically generated

**Ch. 2.5: Induction and Recursion**

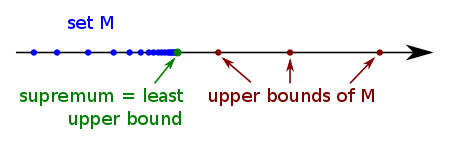
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| **Proposition / Theorem / Lemma / Definition** | **Description** |
| **Principle of Mathematical Induction**  **(Theorem 2.5.1)** | Let G ⊆ ℕ. Suppose that  a. 1 ∈ G;  b. if n ∈ G, then n + 1 ∈ G.  Then G = ℕ. |
| **Proposition 2.5.2** | Example induction proof |
| **Definition 2.5.3** | Let a,b ∈ ℤ.  The set {a, ..., b} is defined by {a, ..., b} = {x ∈ ℤ | a ≤ x ≤ b}. |
| **Principle of Mathematical Induction—Variant/Complete**  **(Theorem 2.5.4)** | Let G ⊆ ℕ. Suppose that  a. 1 ∈ G;  b. if n ∈ ℕ and {1, ..., n} ⊆ G, then n + 1 ∈ G.  Then G = ℕ. |
| **Definition by Recursion**  **(Theorem 2.5.5)** | Let H be a set, let e ∈ H and let k: H → H be a function. Then there is a unique function f: ℕ → H such that f(1) = e, and that f(n + 1) = k(f(n)) for all n ∈ ℕ. |
| **Definition of xn**  **Definition 2.5.6** | Let x ∈ ℝ. The number xn ∈ ℝ is defined for all n ∈ ℕ by letting x1 = x, and xn+1 = x · xn for all x ∈ ℕ. |
| **Lemma 2.5.7** | Let x ∈ ℝ. Suppose that x ≠ 0. Then xn ≠ 0 for all n ∈ ℕ. |
| **Definition: x0**  **Definition 2.5.8** | Let x ∈ ℝ. Suppose that x ≠ 0.  The number x0 ∈ ℝ is defined by x0 = 1.  For each n ∈ ℕ, the number x−n is defined by x−n = (xn)−1. |
| **Power Rules**  **Lemma 2.5.9** | Let x ∈ ℝ, and let n, m ∈ ℤ. Suppose that x ≠ 0.  1. xnxm = xn+m.  2. xn / xm = xn−m. |
| **Polynomial Function**  **Definition 2.5.10** | Let A ⊆ ℝ be a set, and let f: A → ℝ be a function. The function f is a **polynomial function** if there are some n ∈ ℕ ∪ {0} and a0, a1 ,..., an ∈ ℝ such that f(x) = a0 +a1x + ··· + anxn for all x ∈ A. |
| **an+1 = n + an**  **Theorem 2.5.11** | Let H be a set, let e ∈ H and let t: H × ℕ → H be a function. Then there is a unique function g: ℕ → H such that g(1) = e, and that g(n + 1) = t((g(n), n)) for all n ∈ ℕ. |
| **Factorial: n!**  **Example 2.5.12** | We want to define a sequence of real numbers a1, a2, a3 ,... such that a1 = 1, and an+1 = (n + 1)an for all n ∈ ℕ. |
| **max() Function**  **(Example 2.5.13)** |  |
| **Exercise 2.5.3** | Let n ∈ ℕ, and let a1, a2, ..., an ∈ ℝ.  Prove that |a1 + a2 +···+ an| ≤ |a1| + |a2| +···+ |an|. |

**Ch. 2.6: The Least Upper Bound Property**

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| **Theorem / Lemma / Corollary / Definition** | **Description** |
| **Example 2.6.1** | (1) Let A = [3,5). Then 10 is an upper bound of A, and −100 is a lower bound. Hence A is bounded above and bounded below, and therefore A is bounded. |
| **Unique LUB / GLB**  **(Lemma 2.6.2)** | Let A ⊆ ℝ be a non-empty set.  1. If A has a least upper bound, the least upper bound is unique.  2. If A has a greatest lower bound, the greatest lower bound is unique. |
| **lub A / glb A**  **(Definition 2.6.3)** | Let A ⊆ ℝ be a non-empty set.  If A has a least upper bound, it is denoted lub A.  If A has a greatest lower bound, it is denoted glb A. |
| **Least Upper Bound Property**  **(Theorem 1.7.9)** | Let A ⊆ ℝ be a set. If A is nonempty and bounded above, then A has a least upper bound. |
| **Greatest Lower Bound Property**  **(Theorem 2.6.4)** | Let A ⊆ ℝ be a set. If A is non-empty and bounded below, then A has a greatest lower bound. |
| **Lemma 2.6.5** | Let A ⊆ ℝ be a non-empty set, and let ε > 0.  1. Suppose that A has a least upper bound. Then there is some a ∈ A such that lub A − ε < a ≤ lub A.  2. Suppose that A has a greatest lower bound. Then there is some b ∈ A such that glb A ≤ b < glb A + ε. |
| **No Gap Lemma**  **(Lemma 2.6.6)** | Let A,B ⊆ ℝ be non-empty sets. Suppose that if a ∈ A and b ∈ B, then a ≤ b.  1. A has a least upper bound and B has a greatest lower bound, and lub A ≤ glb B.  2. lub A = glb B if and only if for each ε > 0, there are a ∈ A and b ∈ B such that b − a < ε. |
| **Archimedean Property**  **(Theorem 2.6.7)** | Let a,b ∈ ℝ. Suppose that a > 0.  Then there is some n ∈ ℕ such that b < na. |
| ℝ **In-between** ℤ**s**  **(Corollary 2.6.8)** | Let x ∈ ℝ.  1. There is a unique n ∈ ℤ such that n − 1 ≤ x < n. If x ≥ 0, then n ∈ ℕ.  2. If x > 0, there is some m ∈ ℕ such that 1 / m < x. |
| **Square Root**  **Theorem 2.6.9** | Let p ∈ (0,∞). Then there is a unique x ∈ (0,∞) such that x2 = p. |
| **Square Root: √**  **Definition 2.6.10** | Let p ∈ (0,∞). The square root of p, denoted √p, is the unique x ∈ (0,∞) such that x2 = p. |
| √**2 is Irrational**  **(Theorem 2.6.11)** | Let p ∈ ℕ. Suppose that there is no u ∈ ℤ such that p = u2. Then √p ∉ ℚ. |
| ℚ **≠ LUB**  **(Corollary 2.6.12)** | The ordered field ℚ does not satisfy the Least Upper Bound Property. |
| ℝ **Sandwich**  **(Theorem 2.6.13)** | Let a,b ∈ ℝ. Suppose that a < b.  1. There is some q ∈ ℚ such that a < q < b.  2. There is some r ∈ ℝ − ℚ such that a < r < b. |
| **Heine–Borel Theorem**  **(Theorem 2.6.14)** | Let C ⊆ ℝ be a closed bounded interval, let I be a non-empty set and let be a family of open intervals in ℝ. Suppose that . Then there are n ∈ ℕ and i1, i2, ..., in ∈ I such that . |

**Ch. 2.7: Uniqueness of the Real Numbers**

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| **Theorem** | **Description** |
| **Uniqueness of the Real Numbers**  **(Theorem 2.7.1)** | Let R1 and R2 be ordered fields that satisfy the Least Upper Bound Property. Then there is a function f: R1 → R2 that is bijective, and that satisfies the following properties.  Let x,y ∈ R1.  a. f(x + y) = f(x) + f(y).  b. f(xy) = f(x) f(y).  c. If x < y, then f(x) < f(y). |



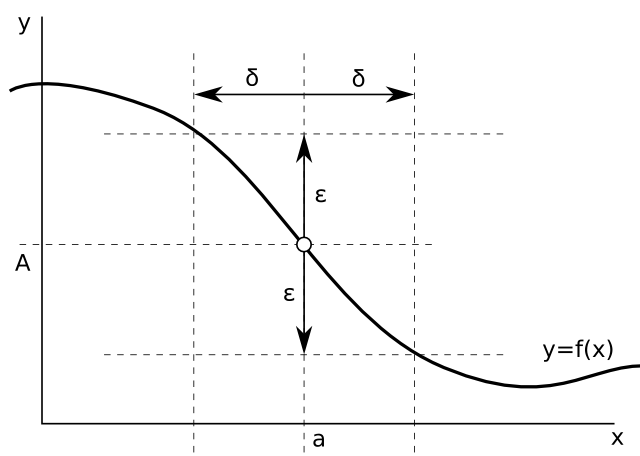
**Ch. 2.8: Decimal Expansion of Real Numbers**

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| **Theorem / Lemma / Definition** | **Description** |
| **Base-p**  **(Lemma 2.8.1)** | Let p ∈ ℕ. Suppose that p > 1. Let n ∈ ℕ. Then there is a unique k ∈ ℕ such that pk−1 ≤ n < pk. |
| **Base-p Numbers**  **(Theorem 2.8.2)** | Let p ∈ ℕ. Suppose that p > 1. Let n ∈ ℕ. Then there are unique k ∈ ℕ and a0, a1, ..., ak−1 ∈ {0, ..., p − 1} such that ak−1 ≠ 0, and that |
| **Base-p Fractions**  **(Lemma 2.8.3)** | Let p ∈ ℕ. Suppose that p > 1. Let a1, a2, a3, ... ∈ {0, ..., p − 1}. Then the set  is bounded below by 0 and is bounded above by 1. [0,1] |
| **Definition 2.8.4** | Let p ∈ ℕ. Suppose that p > 1. Let a1, a2, a3, ... ∈ {0, ..., p − 1}. The sum is defined by |
| **Lemma 2.8.5** | Let p ∈ ℕ. Suppose that p > 1. Let a1, a2, a3, ... ∈ {0, ..., p − 1}.  1. 0 ≤ ≤ 1.  2. if and only if ai = 0 for all i ∈ ℕ.  3. if and only if ai = p − 1 for all i ∈ ℕ.  4. Let m ∈ ℕ. Suppose that m > 1, and that am−1 ≠ p − 1. Then  where equality holds if and only if ai = p − 1 for all i ∈ ℕ such that i ≥ m. |
| **Uniqueness of** ℝ  **(Theorem 2.8.6)** | Let p ∈ ℕ. Suppose that p > 1. Let x ∈ (0,∞).  1. There are k ∈ ℕ, and b0, b1, ..., bk−1 ∈ {0, ..., p − 1} and a1, a2, a3 ... ∈ {0, ..., p − 1}, such that  2. It is possible to choose k ∈ ℕ, and b0, b1, ..., bk−1 ∈ {0, ..., p − 1}, and a1, a2, a3 ... ∈ {0, ..., p − 1} in Part (1) of this theorem such that there is no m ∈ ℕ such that ai = p − 1 for all i ∈ ℕ such that i ≥ m.  3. If x > 1, then it is possible to choose k ∈ ℕ, and b0, b1, ..., bk−1 ∈ {0, ..., p − 1}, and a1, a2, a3 ... ∈ {0, ..., p − 1} in Part (1) of this theorem such that bk−1 ≠ 0.  If 0 < x < 1, then it is possible to choose k = 1, and b0 = 0, and a1, a2, a3 ... ∈ {0, ..., p − 1} in Part (1) of this theorem.  4. If the conditions of Parts (2) and (3) of this theorem hold, then the numbers k ∈ ℕ, and b0, b1, ..., bk−1 ∈ {0, ..., p − 1}, and a1, a2, a3 ... ∈ {0, ..., p − 1} in Part (1) are **unique**. |
| **Base p Representation (bj.ai)**  **(Definition 2.8.7)** | Let p ∈ ℕ. Suppose that p > 1. Let x ∈ (0,∞). A **base p representation** of the number x is an expression of the form x = bk−1 ··· b1b0.a1a2a3 ···, where k ∈ ℕ and b0, b1, ..., bk−1 ∈ {0, ..., p − 1} and a1, a2, a3 ... ∈ {0, ..., p − 1} are such that |
| **Division Algorithm: ÷**  **(Theorem 2.8.8)** | Let a ∈ ℕ ∪ {0} and b ∈ ℕ. Then there are unique q, r ∈ ℕ ∪ {0} such that a = bq + r and 0 ≤ r < b. (q = quotient, r = remainder) |
| **Repeating Decimal**  **(Definition 2.8.9)** | Let p ∈ ℕ. Suppose that p > 1. Let x ∈ (0,∞), and let x = bk−1 ··· b1b0.a1a2a3 ··· be a base p representation of x. This base p representation is **eventually repeating** if there are some r,s ∈ ℕ such that aj = aj+s for all j ∈ ℕ such that j ≥ r; in that case we write  . |
| **Rational if Repeating Decimal**  **(Theorem 2.8.10)** | Let p ∈ ℕ. Suppose that p > 1. Let x ∈ (0,∞). Then x ∈ ℚ if and only if x has an eventually repeating base p representation. |

**Ch. 3.2 Limits of Functions**

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| **Theorem / Lemma / Definition** | **Description** |
| **Limit of a Function**  **(Definition 3.2.1)** | Let I ⊆ ℝ be an open interval, let c ∈ I, let f: I − {c} → ℝ be a function and let L ∈ ℝ. The number L is the **limit** of f as x goes to c, written  if for each ε > 0, there is some δ > 0 such that x ∈ I − {c} and 0 < |x − c| < δ imply |f(x) − L| < ε.  If , we also say that f **converges** to L as x goes to c.  If f converges to some real number as x goes to c, we say that exists.  An **open interval** is an interval that does not include its end points. |
| **Logical Form of Limits** | (∀ε > 0) (∃δ > 0) [(x ∈ I − {c} ∧|x − c| < δ) → |f(x) − L| < ε]  The order of the quantifiers in the definition of limits is absolutely crucial. |
| **Proof Format** | A typical proof that must therefore have the following form:  **Proof**.  Let ε > 0  . . . . (argumentation) . . .  Let δ = f(ε)  . . . (argumentation) . . .  Suppose that x ∈ I − {c} and |x − c| < δ  . . . . (argumentation) . . .  Therefore |f(x) − L| < ε. |
| **L is Unique**  **(Lemma 3.2.2)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I − {c} → ℝ be a function. If for some L ∈ ℝ, then L is unique. |
| **Example Proofs**  **(Example 3.2.3)** | (1) Prove that .  **Proof:**  Let ε > 0. Let δ = ε 5. Suppose that x ∈ ℝ − {4} and |x−4| < δ. Then |(5x + 1) − 21| = |5x − 20| = 5|x − 4| < 5δ = 5 · ε 5 = ε. |
|  | (2) Prove that .  **Proof:**  Let ε > 0. Let δ = min{ε 7 ,1}. Suppose that x ∈ℝ − {3} and |x−3| < δ. **Then** |x−3| < 1, which **implies** that −1 < x−3 < 1, and **therefore** 2 < x < 4, and **hence** 5 < x + 3 < 7, and we **conclude** that 5 < |x + 3| < 7. Then |(x2 − 1) − 8| = |x2 − 9| = |x − 3|·|x + 3| < δ · 7 ≤ ε 7 · 7 = ε. |
|  | (3) Prove that . does not exist.  **Proof:**  Suppose that for some L ∈ ℝ. Let ε = |L| / 2 if L ≠ 0, and let ε = 1 if L = 0. We consider the case when L > 0; the other cases are similar. Let δ > 0. Because L > 0, then L + ε > 0. Let x = min{δ/2, 1 / (L + ε)}. Then x ∈ (0, ∞) and |x − 0| ≤ δ / 2 < δ. On the other hand, because x ≤ 1 / (L + ε), it follows that L + ε ≤ 1 / x, and hence 1 / x − L ≥ ε, which implies that |1 / x − L| ≮ ε. |
| **Sign-Preserving Property for Limits**  **(Theorem 3.2.4)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I − {c} → ℝ be a function. Suppose that exists.  1. If , then there is some M > 0 and some δ > 0 such that x ∈ I − {c} and |x − c| < δ imply f(x) > M.  2. If , then there is some N < 0 and some δ > 0 such that x ∈ I − {c} and |x − c| < δ imply f(x) < N. |
| **Bounded**  **(Lemma 3.2.7)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I − {c} → ℝ be a function. If exists, then there is some δ > 0 such that the restriction of f to (I − {c}) ∩ (c − δ, c + δ) is bounded. |
| **Zero**  **(Lemma 3.2.8)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f,g: I − {c} → ℝ be functions. Suppose that , and that g is bounded. Then . |
| **Functions for +, -, k, •,** ÷  **(Definition 3.2.9)** | Let A,B be sets, let f: A → ℝ and g: B → ℝ be functions and let k ∈ ℝ.  1. The function f + g: A ∩ B → ℝ is defined by [f + g](x) = f(x) + g(x) for all x ∈ A ∩ B.  2. The function f − g: A ∩ B → ℝ is defined by [f − g](x) = f(x) − g(x) for all x ∈ A ∩ B.  3. The function k f: A → ℝ is defined by [k f ](x) = k f(x) for all x ∈ A.  4. The function f · g: A ∩ B → ℝ is defined by [f · g](x) = f(x) · g(x) for all x ∈ A ∩ B.  5. Let C = (A ∩ B) − {b ∈ B | g(b) = 0}. The function f g: C → ℝ is defined by [f/g] (x) = f(x) / g(x) for all x ∈ C.  6. The function | f |: A → ℝ is defined by | f |(x) = | f(x) | for all x ∈ A. |
| **Limits for +, -, k, •,** ÷  **(Theorem 3.2.10)** | Let I ⊆ ℝ be an open interval, let c ∈ I, let f,g: I − {c} → ℝ be functions and let k ∈ ℝ. Suppose that and exist.  1. exists and .  2. exists and .  3. exists and .  4. exists and .  5. exists and if . |
| **Limits for f ∘ g**  **(Theorem 3.2.12)** | Let I,J ⊆ ℝ be open intervals, let c ∈ I, let d ∈ J and let g: I − {c} → J − {d} and f: J − {d} → ℝ be functions. Suppose that and that exist. Then exists, and . |
| **Limits: f ≤ g**  **(Theorem 3.2.13)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f,g: I − {c} → ℝ be functions. Suppose that f(x) ≤ g(x) for all x ∈ I − {c}. If and exist, then . |
| **Squeeze Theorem for Functions**  **(Theorem 3.2.14)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f,g,h: I − {c} → ℝ be functions. Suppose that f(x) ≤ g(x) ≤ h(x) for all x ∈ I − {c}. If for some L ∈ ℝ, then exists and . |

Chart, histogram

Description automatically generated 

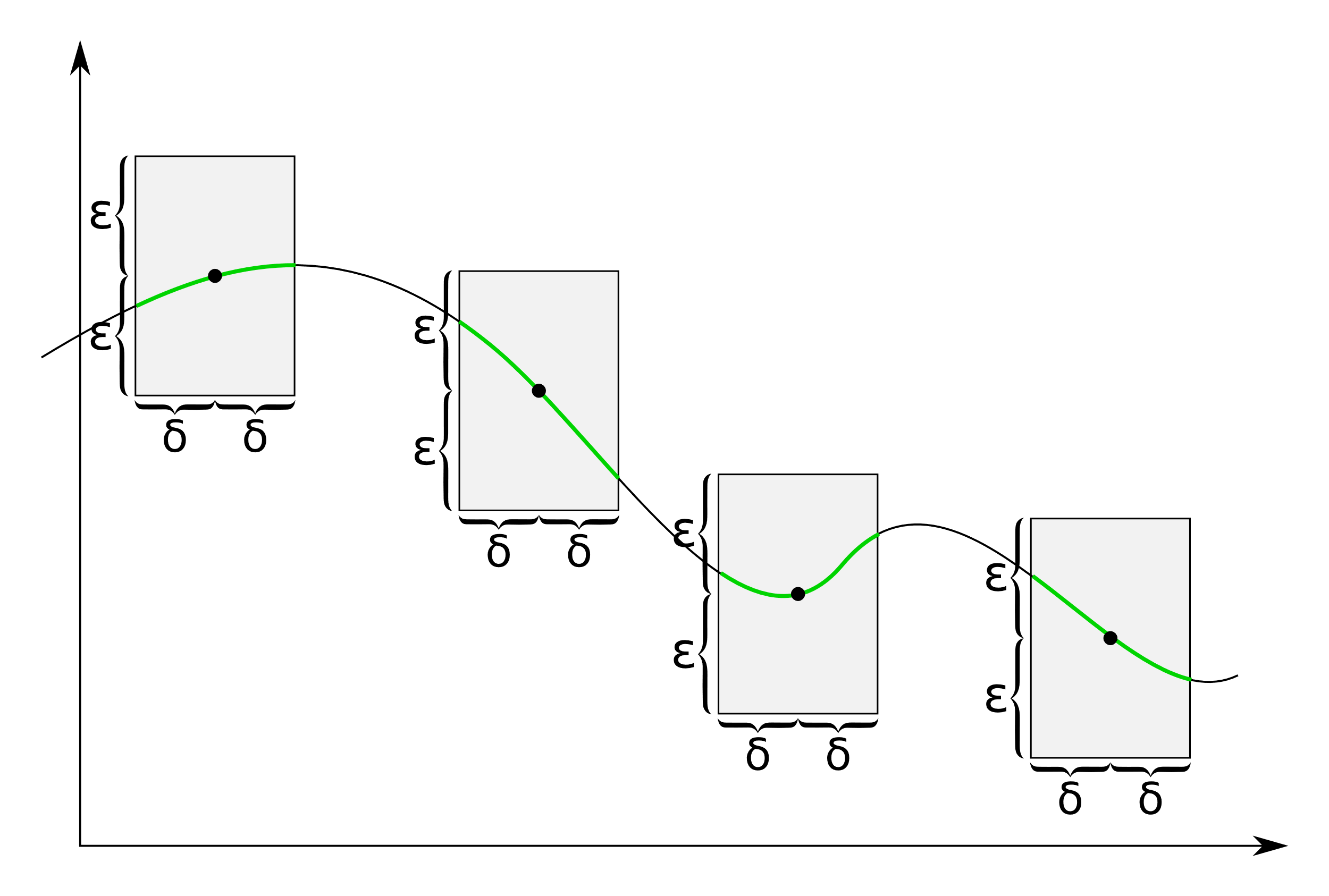
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| **Left/Right Hand Limits**  **(Definition 3.2.15)** | Let I ⊆ ℝ be an interval, let c ∈ I, let f: I − {c} → ℝ be a function and let L ∈ ℝ.  1. Suppose that c is not a right endpoint of I. The number L is the **right-hand limit** of f at c, written  if for each ε > 0, there is some δ > 0 such that x ∈ I − {c} and c < x < c + δ imply |f(x) − L| < ε. If , we also say that f converges to L as x goes to c from the right. If f converges to some real number as x goes to c from the right, we say that exists.  2. Suppose that c is not a left endpoint of I. The number L is the **left-hand limit** of f at c, written  if for each ε > 0, there is some δ > 0 such that x ∈ I − {c} and c − δ < x < c imply |f(x) − L| < ε. If , we also say that f converges to L as x goes to c from the left. If f converges to some real number as x goes to c from the left, we say that  exists.  3. A **one-sided limit** is either a right-hand limit or a left-hand limit. |
| **All 3 Limits are Equal**  **(Lemma 3.2.17)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I − {c} → ℝ be a function. Then exists if and only if and exist and are equal, and if these three limits exist then they are equal. |
| **y = mx + b**  **(Exercise 3.2.1)** | Let m, b, c ∈ ℝ. Using only the definition of limits, prove that |
| **Exercise 3.2.5** | Let J ⊆ I ⊆ ℝ be open intervals, let c ∈ J and let f: I − {c} → ℝ be a function. Prove that exists if and only if exists, and if these limits exist, then they are equal. |

**Ch. 3.3 Continuity**

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| **Theorem / Lemma / Corollary / Definition / Examples** | **Description** |
| **Continuity: ε, δ**  **(Definition 3.3.1)** | Let A ⊆ ℝ be a set, and let f: A → ℝ be a function.  1. Let c ∈ A. The function f is **continuous** at c if for each ε > 0, there is some δ > 0 such that x ∈ A and |x − c| < δ imply |f(x) − f(c)| < ε. The function f is **discontinuous** at c if f is not continuous at c; in that case we also say that f has a discontinuity at c.  2. The function f is continuous if it is continuous **at every number** in A. The function f is discontinuous if it is not continuous. |
| **Continuity: f(c)**  **(Lemma 3.3.2)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I → ℝ be a function. Then f is continuous at c if and only if  exists and . |
| **Logical Form of Continuity** | (∀c ∈ A)[ f is continuous at c]  which can be written completely in symbols as  (∀c ∈ A) (∀ε > 0) (∃δ > 0) [(x ∈ A ∧ |x − c| < δ) → |f(x) − f(c)| < ε].  The order of the quantifiers is crucial.  Applies where we can find δ that depends upon ε and c. |
| **Example 3.3.3** | (1) f(x) = mx + b  (2) p(x) = 1/x  (3) Standard elementary functions (that is, polynomials, power functions, logarithms, exponentials and trigonometric functions). All of these functions are continuous.  (4) y = tan(x)  (5) g(x) = |x|/x  (6) r(x) = 1 or 0  (7) s(x) = 1/q |
| **Sign-Preserving Property for Continuous Functions**  **(Theorem 3.3.4)** | Let A ⊆ ℝ be a non-empty set, let c ∈ A and let f: A → ℝ be a function. Suppose that f is continuous at c.  1. If f(c) > 0, then there is some M > 0 and some δ > 0 such that x ∈ A and |x − c| < δ imply f(x) > M.  2. If f(c) < 0, then there is some N < 0 and some δ > 0 such that x ∈ A and |x − c| < δ imply f(x) < N. |
| **+, -,** ·**, ÷ Continuous at x = c**  **(Theorem 3.3.5)** | Let A ⊆ ℝ be a non-empty set, let c ∈ A, let f,g: A → ℝ be functions and let k ∈ ℝ. Suppose that f and g are continuous at c.  1. f + g is continuous at c.  2. f − g is continuous at c.  3. k · f is continuous at c.  4. f · g is continuous at c.  5. If g(c) ≠ 0, then f/g is continuous at c. |
| **+, -,** ·**, ÷ Continuous Everywhere**  **(Corollary 3.3.6)** | Let A ⊆ ℝ be a non-empty set, let f,g: A → ℝ be functions and let k ∈ ℝ. Suppose that f and g are continuous. Then f + g, f − g, k · f and f · g are continuous, and if g(x) ≠ 0 for all x ∈ I then f / g is continuous. |
| **Example 3.3.7** | (1) fn(x) = xn  (2) p(x) = 1/x |
| **Composite Functions (f ◦ g)**  **(Theorem 3.3.8)** | Let A,B ⊆ ℝ be non-empty sets, let c ∈ A and let g: A → B and f: B → ℝ be functions.  1. Suppose that A is an open interval. If exists and is in B, and if f is continuous at , then = f.  2. If g is continuous at c, and if f is continuous at g(c), then f ◦ g is continuous at c.  3. If g and f are continuous, then f ◦ g is continuous. |
| **Composition of Two Discontinuous Functions**  **(Example 3.3.9)** | (1) h(x) = 1 or 0, k(x) = 2 or 0 m 🡪 Better = Continuous  (2) r(x) = 1 or 0, s(x) = 1/q 🡪 Worse Discontinuity |
| **Pasting Lemma**  **(Lemma 3.3.10)** | Let [a,b] ⊆ ℝ and [b, c] ⊆ ℝ be non-degenerate closed bounded intervals, and let f: [a,b] → ℝ and g: [b, c] → ℝ be functions. Let h: [a,c] → ℝ be defined by h(x) = (f(x), if x ∈ [a,b], g(x), if x ∈ [b, c]. If f and g are continuous, and if f(b) = g(b), then h is continuous. |
| **Extension of a Function**  **(Example 3.3.11)** | f(x) = x 🡪 Can be extended  p(x) = 1/x 🡪 Cannot be extended |

**Ch. 3.4 Uniform Continuity**

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| **Lemma / Corollary / Definition / Examples** | **Description** |
| **Uniformly Continuous (UC)**  **(Definition 3.4.1)** | Let A ⊆ ℝ be a set and let f: A → ℝ be a function. The function f is **uniformly continuous** if for each ε > 0, there is some δ > 0 such that x, y ∈ A and |x − y| < δ imply |f(x) − f(y)| < ε. |
| **Logical Form of UC** | (∀ε > 0) (∃δ > 0) (∀x ∈ A) (∀y ∈ A) [|x − y| < δ → |f(x)− f(y)| < ε]  The order of the quantifiers is crucial.  Applies where we can find δ that depends only upon ε, and not c. |
| **UC 🡪 C**  **(Lemma 3.4.2)** | Let A ⊆ ℝ be a set and let f: A → ℝ be a function. If f is uniformly continuous, then f is continuous. |
| **Example 3.4.3** | (1) f(x) = mx + b 🡪 Is UC  (2) g(x) = 1/x where x ∈ ℝ − {0} 🡪 Is not UC  (3) g(x) = 1/x where x ∈ (1, ∞) 🡪 Is UC |
| **Close Bounded Interval C 🡪 UC**  **(Theorem 3.4.4)** | Let C ⊆ ℝ be a closed bounded interval, and let f: C → ℝ be a function. If f is continuous, then f is uniformly continuous. |
| **UC 🡪 Bounded**  **(Theorem 3.4.5)** | Let A ⊆ ℝ be a non-empty set and let f: A → ℝ be a function. Suppose that A is bounded. If f is uniformly continuous, then f is bounded. |
| **C 🡪 Bounded**  **(Corollary 3.4.6)** | Let C ⊆ ℝ be a closed bounded interval and let f: C → ℝ be a function. If f is continuous, then f is bounded. |

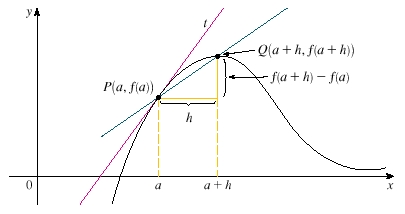


**Ch. 3.5 Two Important Theorems**

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| **Axiom / Theorem / Lemma / Definition** | **Description** |
| **Extreme Value Theorem: Min. and Max. Exist**  **(Theorem 3.5.1)** | Let C ⊆ ℝ be a closed bounded interval and let f: C → ℝ be a function. Suppose that f is continuous. Then there are xmin, xmax ∈ C such that f(xmin) ≤ f(x) ≤ f(xmax) for all x ∈ C. |
| **Intermediate Value Theorem**  **(Theorem 3.5.2)** | Let [a,b] ⊆ ℝ be a closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is continuous. Let r ∈ ℝ. If r is strictly between f(a) and f(b), then there is some c ∈ (a,b) such that f(c) = r. We can assume f(a) < r < f(b). |
| **Contrapositive for a Proof**  **(Lemma 3.5.3)** | Let F be an ordered field. **Suppose that F does not satisfy the Least Upper Bound Property**. Let A ⊆ F be a non-empty set such that A is bounded above, but A has no least upper bound. Let a ∈ A, and let b ∈ F be an upper bound of A. Let Q = {x ∈ [a,b] | x is an upper bound of A} and P = [a,b] − Q.  1. P ∪ Q = [a,b] and P ∩ Q = ∅.  2. a < b, and A ∩ [a,b] ⊆ P, and a ∈ P, and b ∈ Q.  3. If x ∈ P and z ∈ Q, then x < z.  4. If x ∈ P, then there is some y ∈ P such that x < y. If z ∈ Q, then there is some w ∈ Q such that w < z.  5. The set P does not have a least upper bound, and the set Q does not have a greatest lower bound. |
| **Theorem 3.5.4** | The following are equivalent.  a1. The Least Upper Bound Property.  a2. The Greatest Lower Bound Property.  b. The Heine–Borel Theorem.  c. The Extreme Value Theorem.  d. The Intermediate Value Theorem. |

**Ch. 4.2 The Derivative**

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| **Definition / Theorem / Example** | **Description** |
| **Definition of Derivative with x - c**  **(Definition 4.2.1)** | Let be an open interval, let and let be a function.  1. The function f is **differentiable** at c if  exists; if this limit exists, it is called the **derivative** of at , and it is denoted .  2. The function is **differentiable** if it is differentiable at every number in I. If f is differentiable, the **derivative** of is the function f’: I → ℝ whose value at x is f’(x) for all x ∈ I. |
| **Definition of Derivative with h**  **(Lemma 4.2.2)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let : I → ℝ be a function. Then is differentiable at c if and only if  exists, and if this limit exists it equals f’(c). |
| **Example 4.2.3** | (1) f(x) = mx + b so (mx + b)’ = m.  (2) g(x) = x2 so g’(x) = 2x  (3) k(x) = |x| so k’(x) does not exist unless x ∈ (0, ∞). |
| **Differentiable 🡪 Continuous**  **(****Theorem 4.2.4)** | Let I ⊆ ℝ be an open interval, and let f: I → ℝ be a function. Let c ∈ I.  If f is differentiable at c, then f is continuous at c.  If f is differentiable, then f is continuous. |
| **Continuous vs. Differentiable**  **(Example 4.2.5)** | (1) f(x) = {x2 sin (1/x2), if x ≠ 0}  {0, if x = 0}  So, f’ exists everywhere, but f’ is not continuous.  (2) g(x) = { x2, if x ≥ 0}  {- x2, if x < 0}  So, g’ is continuous, however g’ is not differentiable. |
| **nth Derivatives**  **(Definition 4.2.6)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I → ℝ be a function.  Suppose that f is differentiable at c.  The function f is **twice differentiable** at c if f’ is differentiable at c.  If f’ is differentiable at c, the derivative (f’)’(c) is called the **second derivative** of f at c, and it is denoted f’’(c).  The function f is **twice differentiable** if it is twice differentiable at every number in I.  If f is twice differentiable, the second derivative of f is the function f’’: I → ℝ whose value at x is f’’(x) for all x ∈ I.  The nth derivative of f for all n ∈ ℕ is defined as follows, using Definition by Recursion.  If f is differentiable at c, the **first derivative** of f at c is simply the derivative of f at c.  Suppose that f is n−1 times differentiable at c.  The (n−1)-st derivative of f at c is denoted f(n−1)(c).  The function f is **n times differentiable** at c if f(n−1) is differentiable at c.  If f(n−1) is differentiable at c, the derivative (f(n−1) )’(c) is called the **nth derivative** of f at c, and it is denoted f(n)(c).  The function f is n times differentiable if it is n times differentiable at every number in I.  If f is n times differentiable, the nth derivative of f is the function f(n): I → ℝ whose value at x is f(n)(x) for all x ∈ I.  The **0th derivative** of f is f(0) = f. |
| **Continuously/Infinitely Differentiable**  **(Definition 4.2.7)** | Let I ⊆ ℝ be an open interval, and let f: I → ℝ be a function.  The function f is **continuously differentiable** if f is differentiable and f’ is continuous.  Let n ∈ ℕ. The function f is **continuously differentiable of order** n if f(i) exists and is continuous for all i ∈ {1, ..., n}.  The function f is **infinitely differentiable** (also called **smooth**) if f(i) exists all i ∈ ℕ. |
| **One-Sided Derivatives**  **(Definition 4.2.8)** | Let I ⊆ ℝ be a non-degenerate interval, let c ∈ I and let f: I → ℝ be a function.  1. Suppose that c is a left endpoint of I. The function f is **differentiable** at c if the limit  exists; if this limit exists, it is called the **one-sided derivative** of f at c, and it is denoted f’(c).  2. Suppose that c is a right endpoint of I. The function f is **differentiable** at c if the limit  exists; if this limit exists, it is called the **one-sided derivative** of f at c, and it is denoted f’(c).  3. The function f is **differentiable** if the restriction of f to the interior of I is differentiable in the usual sense, and if f is differentiable at the endpoints of I in the sense of Parts (1) and (2) of this definition if there are endpoints. |
| **Symmetric Derivative**  **(Exercise 4.2.7)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I → ℝ be a function. The function f is **symmetrically differentiable** at c if  exists; if this limit exists, it is called the **symmetric** **derivative** of f at c. |



**Ch. 4.3 Computing Derivatives**

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| **Theorem / Corollary** | **Description** |
| **Derivatives: +, −, •, ÷**  **(Theorem 4.3.1)** | Let I ⊆ ℝ be an open interval, let c ∈ I, let f,g: I → ℝ be functions and let k ∈ ℝ. Suppose that f and g are differentiable at c.  1. **f + g** is differentiable at c and [f + g]’(c) = f’(c) + g’(c).  2. **f − g** is differentiable at c and [f − g]’(c) = f’(c) – g’(c).  3. **kf** is differentiable at c and [kf]’(c) = k f’(c).  4. (**Product Rule**) fg is differentiable at c and [fg]’(c) = f’(c)g(c) + f(c)g’(c).  5. (**Quotient Rule**) If g(c) ≠ 0, then f/g is differentiable at c and |
| **Entire Function**  **(Corollary 4.3.2)** | Let I ⊆ ℝ be an open interval, let f,g: I → ℝ be functions and let k ∈ ℝ. If f and g are differentiable, then f + g, f − g, kf and fg are differentiable, and if g(x) ≠ 0 for all x ∈ I then f/g is differentiable. |
| **Chain Rule**  **(Theorem 4.3.3)** | Let I,J ⊆ ℝ be open intervals, let c ∈ I and let f: I → J and g: J → ℝ be functions. Suppose that f is differentiable at c, and that g is differentiable at f(c). Then g ◦ f is differentiable at c and [g ◦ f]’(c) = g’ (f(c))· f’(c). |
| **Chain Rule Differentiable**  **(Corollary 4.3.4)** | Let I,J ⊆ ℝ be open intervals, and let f: I → J and g: J → ℝ be functions. If f and g are differentiable, then g ◦ f is differentiable. |

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**Ch. 4.4 The Mean Value Theorem**

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| **Axiom / Theorem / Lemma / Definition** | **Description** |
| **Min/Max at a Point, f’(c) = 0**  **(Lemma 4.4.1)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let c ∈ (a,b) and let f: [a,b] → ℝ be a function.  Suppose that f is differentiable at c.  If either f(c) ≥ f(x) for all x ∈ [a,b] or f(c) ≤ f(x) for all x ∈ [a,b], then f’(c) = 0. |
| **f’(c) = 0, But Not a Min/Max**  **(Example 4.4.2)** | Let f: [−1,1] → ℝ be defined by f(x) = x3 for all x ∈ [−1,1]. It can be verified using the definition of derivatives that f’(0) = 0; the details are left to the reader. On the other hand, it is certainly not the case that f(0) ≥ f(x) for all x ∈ [−1,1], or that f(0) ≤ f(x) for all x ∈ [−1,1]. |
| **Rolle’s Theorem: f(a) = f(b)**  **(Lemma 4.4.3)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function.  Suppose that f is continuous on [a,b] and differentiable on (a,b).  If f(a) = f(b), then there is some c ∈ (a,b) such that f’(c) = 0.    **Note**: Rolle’s Theorem is a special case of the Mean Value Theorem where f(a) = f(b). |
| **Mean Value Theorem**  **(Average Slope)**  **(Theorem 4.4.4)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function.  Suppose that f is continuous on [a,b] and differentiable on (a,b).  Then there is some c ∈ (a,b) such that    Graphical user interface  Description automatically generated with low confidence  **Note**: The Mean Value Theorem is a special case of Cauchy’s Mean Value Theorem where g(x) = x.  **Note**: The Mean Value Theorem is a special case of Taylor’s Theorem where n = 0, c = a, and x = b |
| **Cauchy’s Mean Value Theorem**  **(Theorem 4.4.5)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f,g: [a,b] → ℝ be functions.  Suppose that f and g are continuous on [a,b] and differentiable on (a,b).  Then there is some c ∈ (a,b) such that  16. AUGUSTIN-LOUIS CAUCHY – SAPAVIVA |
| **Cauchy 🡪 Mean Value Theorem** | The Mean Value Theorem is the special case of Cauchy’s Mean Value Theorem (Theorem 4.4.5) where the function g is defined by g(x) = x for all x ∈ [a,b]. |
| **Taylor’s Theorem**  **(Theorem 4.4.6)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let c ∈ (a,b), let f: [a,b] → ℝ be a function and let n ∈ ℕ ∪ {0}.  Suppose that f(k) exists and is continuous on [a,b] for each k ∈ {0, ..., n}, and that f(n+1) exists on (a,b).  Let x ∈ [a,b].  Then there is some p strictly between x and c (except that p = c when x = c) such that  . |
| **Parallel Functions**  **(Lemma 4.4.7)** | Let I ⊆ ℝ be a non-degenerate interval, and let f,g: I → ℝ be function.  Suppose that f and g are continuous on I and differentiable on the interior of I.  1. f’(x) = 0 for all x in the interior of I if and only if f is constant on I.  2. f’(x) = g’(x) for all x in the interior of I if and only if there is some C ∈ ℝ such that f(x) = g(x) + C for all x ∈ I. |
| **Antiderivative (F’ = f)**  **(Definition 4.4.8)** | Let I ⊆ ℝ be an open interval, and let f: I → ℝ be a function.  An **antiderivative** of f is a function F: I → ℝ such that F is differentiable and F’ = f. |
| **Constant of Integration (+ C)**  **(Corollary 4.4.9)** | Let I ⊆ ℝ be a non-degenerate open interval, and let f: I → ℝ be a function.  If F,G: I → ℝ are antiderivatives of f, then there is some C ∈ ℝ such that F(x) = G(x) + C for all x ∈ I. |
| **Intermediate Value Theorem for Derivatives**  **(Theorem 4.4.10)** | Let I ⊆ ℝ be an open interval, and let f: I → ℝ be a function.  Suppose that f is differentiable.  Let a,b ∈ I, and suppose that a < b.  Let r ∈ ℝ.  If r is strictly between f’(a) and f’(b), then there is some c ∈ (a,b) such that f’(c) = r. |
| **g(x) ≠ f’(x)**  **(Example 4.4.11)** | Let g: ℝ → ℝ be defined by  Then g is not the derivative of any function, because it does not satisfy the conclusion of the Intermediate Value Theorem for Derivatives (Theorem 4.4.10). |

**Ch. 4.5 Increasing and Decreasing Functions, Part I: Local and Global Extrema**

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| **Axiom / Theorem / Lemma / Definition** | **Description** |
| **f(x) vs. Increasing / Decreasing / Monotone**  **(Definition 4.5.1)** | Let A ⊆ ℝ be a set, and let f: A → ℝ be a function.  1. The function f is **increasing** if x < y implies f(x) **≤** f(y) for all x,y ∈ A.  2. The function f is **strictly increasing** if x < y implies f(x) **<** f(y) for all x, y ∈ A.  3. The function f is **decreasing** if x < y implies f(x) **≥** f(y) for all x,y ∈ A.  4. The function f is **strictly decreasing** if x < y implies f(x) **>** f(y) for all x, y ∈ A.  5. The function f is **monotone** if it is either increasing or decreasing.  6. The function f is **strictly monotone** if it is either strictly increasing or strictly decreasing. |
| **f’(x) vs. Increasing**  **(Theorem 4.5.2)** | Let I ⊆ ℝ be a non-degenerate interval, and let f: I → ℝ be a function. Suppose that f is continuous on I and differentiable on the interior of I.  1. f’(x) ≥ 0 for all x in the interior of I if and only if f is **increasing** on I.  2. If f’(x) > 0 for all x in the interior of I, then f is **strictly increasing** on I.  3. f’(x) ≤ 0 for all x in the interior of I if and only if f is **decreasing** on I.  4. If f’(x) < 0 for all x in the interior of I, then f is **strictly decreasing** on I. |
| **Example 4.5.3** | Let f: ℝ → ℝ be defined by f(x) = x3 for all x ∈ ℝ.  The function f is strictly increasing, as seen by Exercise 2.3.3 (1); that exercise does not make use of derivatives.  However, we know by Exercise 4.3.5 that f’(x) = 3x2 for all x ∈ ℝ, and hence f’(0) = 0.  Therefore Theorem 4.5.2 (2) cannot be made into an “if and only if” statement.  A similar example shows that Theorem 4.5.2 (4) cannot be made into an “if and only if” statement. |
| **Local/Global Extremum**  **(Definition 4.5.4)** | Let A ⊆ ℝ be a set, let c ∈ A and let f: A → ℝ be a function.  1. The number c is a **local maximum** of f if there is some δ > 0 such that x ∈ A and |x − c| < δ imply f(x) ≤ f(c).  2. The number c is a **local minimum** of f if there is some δ > 0 such that x ∈ A and |x − c| < δ imply f(x) ≥ f(c).  3. The number c is a **local extremum** of f if it is either a local maximum or a local minimum.  4. The number c is a **global maximum** of f if f(x) ≤ f(c) for all x ∈ A.  5. The number c is a **global minimum** of f if f(x) ≥ f(c) for all x ∈ A.  6. The number c is a **global extremum** of f if it is either a global maximum or a global minimum. |
| **Local Min/Max**  **(Lemma 4.5.5)** | Let A ⊆ ℝ be a set, let c ∈ A and let f: A → ℝ be a function.  1. If there is some δ > 0 such that f|A ∩ (c − δ, c] is increasing and f|A ∩ [c, c + δ) is decreasing, then c is a **local maximum** of f.  2. If there is some δ > 0 such that f|A ∩ (c − δ, c] is decreasing and f|A ∩ [c, c + δ) is increasing, then c is a **local minimum** of f. |
| **Critical Point**  **(Definition 4.5.6)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I → ℝ be a function. The number c is a **critical point** of f if either f is differentiable at c and f 0 (c) = 0, or f is not differentiable at c. |
| **Extremum 🡪 Critical Point**  **(Lemma 4.5.7)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I → ℝ be a function. If c is a **local extremum** of f, then c is a **critical point** of f. |
| **Example 4.5.8** | Let f: [−1,1] → ℝ be defined by f(x) = x3 for all x ∈ [−1,1].  Because f’(x) = 3x2 for all x ∈ ℝ, then f’(0) = 0, and hence 0 is a critical point of f.  However, as remarked in Example 4.5.3, the function f is strictly increasing, and therefore 0 is neither a local maximum nor a local minimum of f. |
| **First Derivative Test**  **(Theorem 4.5.9)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I → ℝ be a function. Suppose that c is a critical point of f, and that f is continuous on I and differentiable on I − {c}.  1. Suppose that there is some δ > 0 such that x ∈ I and c − δ < x < c imply f’(x) ≥ 0, and that x ∈ I and c < x < c + δ imply f’(x) ≤ 0. Then c is a **local maximum** of f.  2. Suppose that there is some δ > 0 such that x ∈ I and c − δ < x < c imply f’(x) ≤ 0, and that x ∈ I and c < x < c + δ imply f’(x) ≥ 0. Then c is a **local minimum** of f.  3. Suppose that there is some δ > 0 such that x ∈ I − {c} and |x − c| < δ imply f’(x) > 0, or that x ∈ I − {c} and |x − c| < δ imply f’(x) < 0. Then c is not a **local extremum** of f. |
| **Second Derivative Test**  **(Theorem 4.5.10)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I → ℝ be a function. Suppose that f is differentiable, that f’(c) = 0 and that f is twice differentiable at c.  1. If f’’(c) > 0, then c is a local minimum of f.  2. If f’(c) < 0, then c is a local maximum of f. |
| **Example 4.5.11** | (1) Let f,g: ℝ → ℝ be defined by f(x) = x3 and g(x) = x4 for all x ∈ ℝ.  It is straightforward to verify that f’(0) = 0 and g’(0) = 0, and that f’’(0) = 0 and g’(0) = 0.  Because x4 = (x2)2 ≥ 0 for all x ∈ ℝ, then 0 is a local (and also global) minimum of g.  As noted in Example 4.5.8, the number 0 is not a local extremum of f. |
|  | (2) Let k: ℝ → ℝ be defined by k(x) = |x| for all x ∈ ℝ.  We saw in Example 4.2.3 (3) that k is not differentiable at 0, and hence 0 is a critical point of k.  We also saw that k’(x) = −1 for all x ∈ (−∞,0), and k’(x) = 1 for all x ∈ (0, ∞).  Because k is not differentiable at 0, we cannot apply the Second Derivative Test (Theorem 4.5.10) to k at 0.  However, the First Derivative Test (Theorem 4.5.9) can still be applied, and we see that 0 is a local minimum of k, which is just what we would expect by looking at the graph of k. |
| **Local 🡪 Global**  **(Theorem 4.5.12)** | Let I ⊆ ℝ be an open interval, let c ∈ I and let f: I → ℝ be a function. Suppose that f is continuous, and that c is the **only critical point** of f.  1. If c is a local maximum, then it is a global maximum.  2. If c is a local minimum, then it is a global minimum. |

**Ch. 4.6 Increasing and Decreasing Functions, Part II: Further Topics**

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| **Axiom / Theorem / Lemma / Definition** | **Description** |
| **Not Differentiable**  **(Example 4.6.1)** | Let f: ℝ → ℝ be defined by f(x) = x3 for all x ∈ ℝ.  Intuitively, we know that the function f is bijective, and hence it has an inverse function f−1: ℝ → ℝ, which we write as f−1(x) = for all x ∈ ℝ.  Moreover, we know that the graph of f−1 is obtained from the graph of f by reflection in the line y = x.  Because f has a horizontal tangent line at the origin, then the graph of f−1 has a vertical tangent line at x = 0, which makes it not differentiable at x = 0. |
| **Bounded Intervals**  **(Lemma 4.6.2)** | Let I ⊆ ℝ be a non-degenerate open interval and let f: I → ℝ be a function. Suppose that f is **strictly monotone**.  1. The function f: I → f(I) is **bijective**.  2. Suppose that f is continuous. Then f(I) is a non-degenerate open interval, and one of the following holds:  a. If the interval f(I) is bounded, then f(I) = (glb f(I),lub f(I)).  b. If the interval f(I) is bounded above but is not bounded below, then f(I) = (−∞,lub f(I)).  c. If the interval f(I) is bounded below but is not bounded above, then f(I) = (glb f(I), ∞).  d. If the interval f(I) is not bounded above and is not bounded below, then f(I) = ℝ. |
| **Example 4.6.3** | We want to show that the square root function is continuous.  Let f: (0, ∞) → ℝ be defined by f(x) = x2 for all x ∈ ℝ.  By Exercise 3.5.6 (1) we see that f is strictly increasing, and by Example 3.3.7 (1) we see that f is continuous.  Exercise 3.5.6 implies that f((0, ∞)) = (0, ∞).  It then follows from Lemma 4.6.2 (3) that f−1: (0, ∞) → (0, ∞) is continuous and strictly increasing.  By Definition 2.6.10 we see that f−1(x) = for all x ∈ (0, ∞).  The continuity of this function could also be shown directly by an ε–δ proof, but Lemma 4.6.2 allows us to avoid that. |
| **Inverse Derivatives**  **(Theorem 4.6.4)** | Let I ⊆ ℝ be a non-degenerate open interval, and let f: I → ℝ be a function. Suppose that f is differentiable, and that f’(x) ≠ 0 for all x ∈ I.  1. The function f is strictly monotone.  2. The function f: I → f(I) is bijective.  3. The function f−1: f(I) → I is differentiable.  4. The derivative of f−1 is given by  for all x ∈ f(I). |
| **Secant Line**  **(Definition 4.6.5)** | Let I ⊆ ℝ be an open interval, let a,b ∈ I and let f: I → ℝ be a function. Suppose that a < b.  The **secant line** through (a, f(a)) and (b, f(b)) is the function Sa,b: ℝ → ℝ defined by  for all x ∈ ℝ.  The slope of the secant line through (a, f(a)) and (b, f(b)), denoted Ma,b, is defined by |
| **Function vs Secant Line**  **(Theorem 4.6.6)** | Let I ⊆ ℝ be an open interval, and let f: I → ℝ be a function. The following are equivalent.  a. If a,b ∈ I and a < b, then f(x) ≤ Sa,b(x) for all x ∈ [a,b]  (Function Lies Below Its Secant Lines).  b. If a,b,c ∈ I and a < b < c, then Ma,b ≤ Mb,c  (Function Has Increasing Secant Line Slopes). |
| **Concave Up**  **(Definition 4.6.7)** | Let I ⊆ ℝ be an open interval, and let f: I → ℝ be a function. The function f is **concave up** if either of the two conditions in Theorem 4.6.6 hold. |
| **Theorem 4.6.8** | Let I ⊆ ℝ be an open interval, and let f: I → ℝ be a function.  1. Suppose that f is differentiable. Then the two conditions in Theorem 4.6.6 hold if and only if f’ is increasing on I.  2. Suppose that f is twice differentiable. Then the two conditions in Theorem 4.6.6 hold if and only if f’’(x) ≥ 0 for all x ∈ I. |

**Ch. 5.2 The Riemann Integral**

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| **Axiom / Theorem / Lemma / Definition** | **Description** |
| **Definition 5.2.1** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval.  1. A **partition** of [a,b] is a set P = {x0, x1 , ..., xn} such that a = x0 < x1 < ··· < xn = b, for some n ∈ ℕ.  2. If P = {x0, x1, ..., xn} is a partition of [a,b], the **norm** (also called the **mesh**) of P, denoted ||P||, is defined by  ||P|| = max{x1 − x0, x2 − x1, ..., xn − xn−1}.  3. If P = {x0, x1, ..., xn} is a partition of [a,b], a **representative set** of P is a set T = {t1, t2, ..., tn} such that ti ∈ [xi−1, xi] for all i ∈ {1, ..., n}. |
| **Riemann Sum**  **(Definition 5.2.2)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let f: [a,b] → ℝ be a function, let P = {x0, x1, ..., xn} be a partition of [a,b] and let T = {t1, t2, ..., tn} be a representative set of P. The **Riemann sum** of f with respect to P and T, denoted S(f, P, T), is defined by  Suma de Riemann |
| **Example 5.2.3** | (1) f(x) = x2 |
|  | (2) r(x) = {1 or 0} |
| **Definition of Integrability (ε- δ)**  **(Definition 5.2.4)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let f: [a,b] → ℝ be a function and let K ∈ ℝ. The number K is the **Riemann integral** of f, written  if for each ε > 0, there is some δ > 0 such that if P is a partition of [a,b] with ||P|| < δ, and if T is a representative set of P, then |S(f, P, T) − K| < ε. If the Riemann integral of f exists, we say that f is **Riemann integrable**. |
| **Unique K**  **(Lemma 5.2.5)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function.  If f is Riemann integrable, then there is a unique K ∈ ℝ such that |
| **Example 5.2.6** | (1) f(x) = c |
| (2) g(x) = {7 or 0} |
| (3) r(x) = {0 or 1} |
| (4) s(x) = {1/q or 0} |
| (5) v(x) = {0 or 1} |
| **Exercise 5.2.1** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let ε > 0. Prove that there is a partition R of [a,b] such that ||R|| < ε. |

**Ch. 5.3 Elementary Properties of the Reimann Integral**

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| **Axiom / Theorem / Lemma / Definition** | **Description** |
| **Integration: +, -, k**  **(Theorem 5.3.1)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let f,g: [a,b] → ℝ be functions and let k ∈ ℝ. Suppose that f and g are integrable.  1. f + g is integrable and  2. f − g is integrable and  3. k·f is integrable and  4. |
| **Theorem 5.3.2** | Let [a,b] ⊆ ℝ be a closed bounded interval, and let f,g: [a,b] → ℝ be functions. Suppose that f and g are integrable.  1. If f(x) ≥ 0 for all x ∈ [a,b], then  2. If f(x) ≥ g(x) for all x ∈ [a,b], then  3. Let m, M ∈ ℝ. If m ≤ f(x) for all x ∈ [a,b], then m(b−a) ≤ , and if f(x) ≤ M for all x ∈ [a,b], then . |
| **Integrable 🡪 Bounded**  **(Theorem 5.3.3)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. If f is **integrable**, then f is **bounded**. |

**Ch. 5.4 Upper Sums and Lower Sums**

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| **Axiom / Theorem / Lemma / Definition** | **Description** |
| **Refinement**  **(Definition 5.4.1)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let P and Q be partitions of [a,b]. The partition Q is a **refinement** of P if P ⊆ Q. |
| **Example 5.4.2** | The sets P = {0, ½ , 1}, and Q = {0, ¼, ½, ¾, 1} and ℝ = {0, 1/3, 2/3 , 1} are partitions of [0,1]. Then Q is a refinement of P, but ℝ is not a refinement of P. |
| **Norm of a Refinement**  **(Lemma 5.4.3)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let P and Q be partitions of [a,b].  1. P ∪ Q is a partition of [a,b], and P ∪ Q is a refinement of each of P and Q.  2. If Q is a refinement of P, then ||Q|| ≤ ||P||. |
| **Upper/Lower Sums**  **(Definition 5.4.4)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let f: [a,b] → ℝ be a function and let P = {x0, x1, ..., xn} be a partition of [a,b]. Suppose that f is bounded.  1. For each i ∈ {1, ..., n}, let Mi(f) = lub f([xi−1, xi]) and mi(f) = glb f([xi−1, xi]). If it is necessary to indicate the partition being used, we will write and .  2. The **upper sum** of f with respect to P, denoted U(f, P), is defined by  and the **lower sum** of f with respect to P, denoted  NOTE: An **upper sum** of a continuous function, *f*, takes a point ci in each subinterval where the maximum value of *f* is achieved.  A **lower sum** takes the minimum value of *f* for each subinterval.  Diagram  Description automatically generated with medium confidence  Diagram  Description automatically generated |
| **Example 5.4.5** | (1) f(x) = x2 |
|  | (2) g(x) = {7 or 0} |
|  | (3) r(x) = {1 or 0} |
| **Lemma 5.4.6** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let f: [a,b] → ℝ be a function and let P be a partition of [a,b]. Suppose that f is bounded.  1. If T is a representative set of P, then L(f, P) ≤ S(f, P,T ) ≤ U(f, P).  2. If ℝ is a refinement of P, then L(f, P) ≤ L(f, ℝ) ≤ U(f, ℝ) ≤ U(f, P).  3. If Q is a partition of [a,b], then L(f, P) ≤ U(f, Q). |
| **Integrable Equivalents**  **(Theorem 5.4.7)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is bounded. The following are equivalent.  a. The function f is integrable.  b. For each ε > 0, there is some δ > 0 such that if P is a partition of [a,b] with ||P|| < δ, then U(f, P) − L(f, P) < ε.  c. For each ε > 0, there is some partition P of [a,b] such that U(f, P) − L(f, P) < ε. |
| **Upper/Lower Integral**  **(Definition 5.4.8)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is bounded.  The **upper integral** of f, denoted , is defined by  and the **lower integral** of f, denoted , is defined by |
| **Lemma 5.4.9** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is bounded. Then the upper integral and lower integral of f always exist, and |
| **Proper Integral**  **(Theorem 5.4.10)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is bounded. Then f is integrable if and only if  and if this equality holds then |
| **Continuous 🡪 Integrable**  **(Theorem 5.4.11)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. If f is continuous, then f is integrable. |

**Ch. 5.5 Further Properties of the Reimann Integral**

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| **Axiom / Theorem / Lemma / Definition** | **Description** |
| **g ◦ f is Integrable**  **(Theorem 5.5.1)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let D ⊆ ℝ be a set and let f: [a,b] → ℝ and g: D → ℝ be functions. Suppose that f is integrable, and that f([a,b]) ⊆ D.  1. If g is uniformly continuous and bounded, then g ◦ f is integrable.  2. If D is a non-degenerate closed bounded interval and g is continuous, then g ◦ f is integrable. |
| **Example 5.5.2** | Let f,g: [0,1] → ℝ be defined by f(x) = 1 for all x ∈ [0,1], and g(x) = (1, if x = 0 x, if x ∈ (0,1]. Then (f/g)(x) = (1, if x = 0, 1/x, if x ∈ (0,1].  We know by Example 5.2.6 (1) that f is integrable. The function g is also integrable, as can be seen by combining Exercise 5.2.6 and Exercise 5.3.3 (3). However, even though g(x) ≠ 0 for all x ∈ [0,1], the function f g is not integrable, because integrable functions are bounded by Theorem 5.3.3, and yet f g is not bounded, a fact that is evident by looking at the graph of f g, and is proved in Example 3.2.6. |
| **Definition 5.5.3** | Let A ⊆ ℝ be a set, and let f: A → ℝ be a function. The function f is **bounded away from zero** if there is some P > 0 such that |f(x)| ≥ P for all x ∈ A. |
| **What is Integrable**  **(Theorem 5.5.4)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f,g: [a,b] → ℝ be functions. Suppose that f and g are integrable. 1. fn is integrable for all n ∈ ℕ.  2. fg is integrable.  3. If g is bounded away from zero, then f/g is integrable. |
| **Absolute Value of Integral**  **(Theorem 5.5.5)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. If f is integrable, then |f| is integrable and |
| **Theorem 5.5.6** | Let D ⊆ C ⊆ ℝ be non-degenerate closed bounded intervals, and let f: C → ℝ be a function. If f is integrable, then f|D is integrable. |
| **Intermediate Bound**  **(Theorem 5.5.7)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, let c ∈ (a,b) and let f: [a,b] → ℝ be a function.  1. f is integrable if and only if f|[a,c] and f|[c, b] are integrable.  2. If f is integrable, then |
| **Swap Bounds / Same Bounds**  **(Definition 5.5.8)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is integrable.  Let be defined by  and let be defined by |
| **Split Bounds of Integration**  **(Corollary 5.5.9)** | Let C ⊆ ℝ be a closed bounded interval, and let f: C → ℝ be a function. Let a,b, c ∈ C. If f is integrable, then |

**Ch. 5.6 Fundamental Theorem of Calculus**

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| **Axiom / Theorem / Lemma / Definition** | **Description** |
| **Example 5.6.1** | (1) Let f: [0,2] → ℝ be defined by f(x) = x for all x ∈ [0,2]. Let F: [0,2] → ℝ be defined by |
| (2) Let h: [0,2] → ℝ be defined by h(x) = (1, if x ∈ [0,1] 2, if x ∈ (1,2]. |
| **Fundamental Theorem of Calculus Version I**  **(Theorem 5.6.2)** | Let I ⊆ ℝ be a non-degenerate interval, let a ∈ I and let f: I → ℝ be a function. Suppose that f|C is integrable for every non-degenerate closed bounded interval C ⊆ I. Let F: I → ℝ be defined by  for all x ∈ I.  Let c ∈ I.  If f is continuous at c, then F is differentiable at c and F’(c) = f(c).  If f is continuous, then F is differentiable and F’ = f. |
| **Continuous 🡪 Antiderivative**  **(Corollary 5.6.3)** | Let I ⊆ ℝ be a non-degenerate interval, and let f: I → ℝ be a function.  If f is continuous, then f has an antiderivative. |
| **Fundamental Theorem of Calculus Version II**  **(Theorem 5.6.4)** | Let [a,b] ⊆ ℝ be a non-degenerate closed bounded interval, and let f: [a,b] → ℝ be a function. Suppose that f is integrable and f has an antiderivative. If F: [a,b] → ℝ is an antiderivative of f, then |
| **Example 5.6.5** | (1) Let f: ℝ → ℝ be defined by f(x) = (x2 sin 1/x2, if x ≠ 0 0, if x = 0. |
|  | (2) Let h: [0,2] → ℝ be defined by h(x) = (1, if x ∈ [0,1], 2, if x ∈ (1,2]. |
| **Example 5.6.6** | (1) Let g: [0,2] → ℝ be defined by g(x) = x2 for all x ∈ [0,2]. |
|  | (2) |

**Sources**:

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* [SNHU MAT 470](https://www.snhu.edu/admission/academic-catalogs/coce-catalog#/courses/VydU8ZIYx) - Real Analysis, [The Real Numbers and Real Analysis](https://www.amazon.com/gp/product/0387721762/ref=ox_sc_act_title_4?smid=A1C79WJQJ5SBBJ&psc=1), Ethan D. Bloch, Springer New York, 2011.