Harold's Partial Fraction Decomposition (Calculus)
Cheat Sheet

9 February 2025

Partial Fractions	(http://en.wikipedia.org/wiki/Partial_fraction_decomposition)
Condition	$f(x) = \frac{P(x)}{Q(x)} = \frac{ax^n + \cdots + b}{cx^m + \cdots + d}$ where $P(x)$ and $Q(x)$ are polynomials
Preparation	Case 1: $n \geq m$, Perform long division first Case 2: $n < m$, Proceed to the cases below
Case I: Simple linear (1^{st} degree)	$\frac{A}{(ax+b)}$ or $\frac{A}{x}$
Case II: Multiple degree linear (1^{st} degree)	$\frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}$
Case III: Simple quadratic (2^{nd} degree)	$\frac{Ax+B}{(ax^2+bx+c)}$
Case IV: Multiple degree quadratic (2^{nd} degree)	$\frac{Ax+B}{(ax^2+bx+c)} + \frac{Cx+D}{(ax^2+bx+c)^2} + \frac{Ex+F}{(ax^2+bx+c)^3}$

Example Expansion	$\frac{P(x)}{(ax+b)(cx+d)^2(ex^2+fx+g)}$ $= \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2} + \frac{Dx+E}{(ex^2+fx+g)}$

Typical Solution for Cases I & II	$\int \frac{a}{x+b}\,dx = a\,\ln\lvert x+b\rvert + C$
Typical Solution for Cases III & IV	$\int \frac{a}{x^2+b^2}\,dx = \frac{a}{b}\,tan^{-1}\left(\frac{x}{b}\right) + C$

Steps to Solve	Calculus Example
1. Write down problem	$\int \frac{5x+1}{2x^2-x-1}\,dx$
2. Check if long division is needed	Not needed since degree of numerator (top) is less than degree of denominator (bottom)
3. Factor the denominator	$\frac{5x+1}{(2x+1)(x-1)}$
4. Expand function with A, B, Cs	$\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{(2x+1)} + \frac{B}{(x-1)}$
5. Find a common denominator	$= \frac{A(x-1)}{(2x+1)(x-1)} + \frac{B(2x+1)}{(2x+1)(x-1)}$
6. Focus on numerator	$5x+1 = A(x-1) + B(2x+1)$

7. FOIL if necessary	$(x+1)(x-2) = x^2 - x - 2$
8. Expand/distribute the A, B, Cs	$5x + 1 = Ax - A + 2Bx + B$
9. Regroup by powers of x. (x^2, x, c)	$5x + 1 = Ax + 2Bx - A + B$
10. Factor by powers of x. $()x^2 + ()x + (c)$	$(5)x + (1) = (A + 2B)x + (-A + B)$
11. Introduce ghost factors if needed (0, 1)	$5x + 1 = (0)x^2 + (5)x + (1)$
12. Match left and right coefficients for a system of equations	$A + 2B = 5$ $-A + B = 1$
13. Solve system of equations	Pick simplest method below
a. Substitution method	$B = A + 1$ $A + 2(A+1) = 5$ $A + 2A + 2 = 5$ $3A = 3$ $A = 1$ $B = A + 1 = 1 + 1 = 2$ $\boldsymbol{A = 1, B = 2}$
b. Row elimination method	$A + 2B = 5$ $+[-A + B = 1]$ ------------------------- $3B = 6$ $\boldsymbol{B = 2}$ $A + 2B = 5$ $-2\,[-A + B = 1]$ ------------------------- $3A = 3$ $\boldsymbol{A = 1}$
c. Augmented matrix method	$\left[\begin{matrix} A & B \\ A & B \end{matrix}\middle\vert\begin{matrix} k \\ k \end{matrix}\right] = \left[\begin{matrix} 1 & 2 \\ -1 & 1 \end{matrix}\middle\vert\begin{matrix} 5 \\ 1 \end{matrix}\right]$ Use TI-84 $rref()$ function $= \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle\vert\begin{matrix} 1 \\ 2 \end{matrix}\right]$ $\boldsymbol{A = 1, B = 2}$
14. Reassemble newly expanded function	$\dfrac{5x+1}{(2x+1)(x-1)} = \dfrac{1}{(2x+1)} + \dfrac{2}{(x-1)}$
15. Verify function for accuracy	*Verify the two equations are the same by plugging in any value for x and see if f(x) is the same for both.*
16. Restate problem with expanded function	$\displaystyle\int \frac{1}{2x+1}\,dx + \int \frac{2}{x-1}\,dx$
17. Integrate restated problem	$\displaystyle\int \frac{1}{2x+1}\,dx$ $u = 2x + 1$ $du = 2\,dx$ $\displaystyle\frac{1}{2}\int \frac{1}{u}\,2\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln\lvert u\rvert = \frac{1}{2}\ln\lvert 2x+1\rvert$ $= \dfrac{1}{2}\ln\lvert 2x+1\rvert + 2\ln\lvert x-1\rvert + C$
18. Simplify	$= \ln\sqrt{\lvert 2x+1\rvert} + \ln(x-1)^2 + C$
19. DONE	$= \ln\left[\sqrt{\lvert 2x+1\rvert}\,(x-1)^2\right] + C$