Harold's Infinite Series
Cheat Sheet

22 September 2025

Summation Form	Expanded Form
Exponential Functions	
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots$
$a^x = e^{x \ln(a)} = \sum_{n=0}^{\infty} \frac{[x \ln(a)]^n}{n!}$ for all x	$1 + x \ln(a) + \frac{[x \ln(a)]^2}{2!} + \frac{[x \ln(a)]^3}{3!} + \cdots$

Natural Logarithm Functions	
$\ln(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ for $\lvert x \rvert < 1$	$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \cdots$
$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$ for $\lvert x \rvert < 1$	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$
$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ for $\lvert x \rvert < 1$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \cdots$
$\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=1}^{\infty} \frac{2}{2n-1} x^{2n-1}$ for $\lvert x \rvert < 1$	$2x - \frac{2x^2}{3} + \frac{2x^3}{5} - \frac{2x^4}{7} + \frac{2x^5}{9} - \frac{2x^6}{11} + \frac{2x^7}{13} - \cdots$

Geometric Series	
$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$ for $0 < x < 2$	$1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \cdots$
$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ for $\lvert x \rvert < 1$	$1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - \cdots$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $\lvert x \rvert < 1$	$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \cdots$
$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $\lvert x \rvert < 1$	$1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} - x^{14} + \cdots$
$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$ for $\lvert x \rvert < 1$	$1 + x^2 + x^4 + x^6 + x^8 + x^{10} + x^{12} + x^{14} + \cdots$
$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n-1}\, n x^{n-1}$ for $\lvert x \rvert < 1$	$1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - \cdots$

$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ for $\lvert x \rvert < 1$	$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \cdots$
$\frac{1}{(1+x)^3} = \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\,(n-1)n}{2} x^{n-2}$ for $\lvert x \rvert < 1$	$1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + 28x^6 - \cdots$
$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{(n-1)n}{2} x^{n-2}$ for $\lvert x \rvert < 1$	$1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + \cdots$
$\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n)!}{4^n\,(n!)^2\,(1-2n)} x^n$ for $-1 < x \le 1$	$1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1{,}024}x^5 + \cdots$
$\sqrt{1-x} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n\,(n!)^2\,(1-2n)} x^n$ for $-1 < x \le 1$	$1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \frac{7}{256}x^5 - \frac{21}{1{,}024}x^5 - \cdots$
$\sqrt{1+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n)!}{4^n\,(n!)^2\,(1-2n)} x^{2n}$ for $-1 < x \le 1$	$1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \frac{5}{128}x^8 + \frac{7}{256}x^{10} - \cdots$
$\sqrt{1-x^2} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n\,(n!)^2\,(1-2n)} x^{2n}$ for $-1 < x \le 1$	$1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \frac{7}{256}x^{10} - \cdots$
Double Factorial (!!)	$(n)!! = n(n-2)(n-4)\ldots 6\cdot 4\cdot 2\ if\ even$ $(n)!! = n(n-2)(n-4)\ldots 5\cdot 3\cdot 1\ if\ odd$ where $0!! = 1$ and $-1!! = 1$
$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n-1)!!}{(2n)!!} x^n$ for $-1 < x \le 1$	$1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 - \cdots$
$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^n$ for $-1 < x \le 1$	$1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 + \cdots$
$\frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n-1)!!}{(2n)!!} x^{2n}$ for $-1 < x \le 1$	$1 - \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8 - \cdots$
$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^{2n}$ for $-1 < x \le 1$	$1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8 + \cdots$

Binomial Series	
$$(1+x)^r = \sum_{n=0}^{+\infty} \binom{r}{n} x^n$$ for $\lvert x\rvert < 1$ and all complex r where $$\binom{r}{n} = \frac{n!}{r!\,(n-r)!} = \prod_{k=1}^{n} \frac{r-k+1}{k} = \frac{r(r-1)(r-2)\ldots(r-n+1)}{n!}$$	$1 + rx + \frac{r(r-1)}{2!} x^2 + \frac{r(r-1)(r-2)}{3!} x^3 + \cdots$

Trigonometric Functions	
$\mathbf{\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$ **for all x**	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \cdots$
$\mathbf{\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}$ **for all x**	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!} + \cdots$
$\tan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 2^{2n}\,(2^{2n}-1)\, B_{2n}}{(2n)!} x^{2n-1}$ for $\lvert x\rvert < \frac{\pi}{2}$	$x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2{,}835}x^9 + \frac{1{,}382}{155{,}925}x^{11} + \frac{21{,}844}{608{,}1075}x^{13} + \cdots$
$\sec(x) = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}$ for $\lvert x\rvert < \frac{\pi}{2}$	$1 + \frac{x^2}{2!} + 5\frac{x^4}{4!} + 61\frac{x^6}{6!} + 1{,}385\frac{x^8}{8!} + 50{,}521\frac{x^{10}}{10!} + \cdots$
$\csc(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}\, 2\,(2^{2n-1}-1)\, B_{2n}}{(2n)!} x^{2n-1}$ for $0 < x < \pi$	$\frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15{,}120}x^5 + \frac{127}{604{,}800}x^7 + \cdots$
$\cot(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n}\, B_{2n}}{(2n)!} x^{2n-1}$ for $0 < x < \pi$	$\frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{189}x^5 - \frac{1}{4{,}725}x^7 - \frac{4}{2{,}835}x^9 - \cdots$

Inverse Trigonometric Functions	
$\sin^{-1}(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2 (2n+1)} x^{2n+1}$ for $\lvert x\rvert \le 1$ $\sin^{-1}(x) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\, n!} x^{2n+1}$	$x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3x^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \frac{1\cdot 3\cdot 5\cdot 7x^9}{2\cdot 4\cdot 6\cdot 8\cdot 9} + \cdots$
$\cos^{-1}(x) = \frac{\pi}{2} - \sin^{-1}(x)$ for $\lvert x\rvert \le 1$	$\frac{\pi}{2} - x - \frac{x^3}{2\cdot 3} - \frac{1\cdot 3x^5}{2\cdot 4\cdot 5} - \frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} - \cdots$

$\boldsymbol{\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1}}$ **for** $\boldsymbol{\lvert x \rvert < 1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \; for -1 < x < 1$ $\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \frac{1}{9x^9} + \cdots \; for \; x \geq 1$ $-\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \frac{1}{9x^9} + \cdots \; for \; x < 1$
$\sec^{-1}(x) = -i\ln(x) + i\ln(2) - \frac{i}{4}\sum_{n=0}^{\infty} \frac{(2n+1)! \; x^{2n+2}}{4^n[(n+1)!]^2}$	$-i\ln(x) + i\ln(2) - \frac{i}{4}x^2 - \frac{3i}{32}x^4 - \frac{5i}{96}x^6 - \cdots$
$\csc^{-1}(x) = i\ln(x) - i\ln(2) + \frac{\pi}{2} + \frac{i}{4}\sum_{n=0}^{\infty} \frac{(2n+1)! \; x^{2n+2}}{4^n[(n+1)!]^2}$	$i\ln(x) - i\ln(2) + \frac{\pi}{2} + \frac{i}{4}x^2 + \frac{3i}{32}x^4 + \frac{5i}{96}x^6 + \cdots$
$\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x)$ for $\lvert x \rvert < 1$	$\frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^9}{9} + \cdots \; for -1 < x < 1$ $\frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \frac{1}{9x^9} - \cdots \; for \; x \geq 1$ $\pi + \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \frac{1}{9x^9} - \cdots \; for \; x < 1$

Hyperbolic Functions	
$\boldsymbol{\sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}}$ **for all** $\boldsymbol{x}$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \frac{x^{15}}{15!} + \cdots$
$\boldsymbol{\cosh(x) = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$ **for all** $\boldsymbol{x}$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \frac{x^{14}}{14!} + \cdots$
$\tanh(x) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)\,B_{2n}}{(2n)!} x^{2n-1}$ for $\lvert x \rvert < \frac{\pi}{2}$	$x - 2\frac{x^3}{3!} + 16\frac{x^5}{5!} - 272\frac{x^7}{7!} + 7{,}936\frac{x^9}{9!} - 353{,}792\frac{x^{11}}{11!} + \cdots$ $x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{62}{2{,}835}x^9 - \frac{1{,}382}{155{,}925}x^{11} + \cdots$
$\operatorname{sech}(x) = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} x^{2n}$ for $\lvert x \rvert < \frac{\pi}{2}$	$1 - \frac{x^2}{2!} + 5\frac{x^4}{4!} - 61\frac{x^6}{6!} + 1{,}385\frac{x^8}{8!} - 50{,}521\frac{x^{10}}{10!} + \cdots$
$\operatorname{csch}(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(2-2^{2n})\,B_{2n}}{(2n)!} x^{2n-1}$ for $0 < \lvert x \rvert < \pi$	$\frac{1}{x} - \frac{1}{6}x + \frac{7}{360}x^3 - \frac{31}{15{,}120}x^5 + \frac{127}{604{,}800}x^7 - \cdots$
$\coth(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,2^{2n}\,B_{2n}}{(2n)!} x^{2n-1}$ for $0 < \lvert x \rvert < \pi$	$\frac{1}{x} + \frac{1}{3}x - \frac{1}{45}x^3 + \frac{2}{945}x^5 - \frac{1}{4{,}725}x^7 + \frac{4}{2{,}835}x^9 - \cdots$

Inverse Hyperbolic Functions	
$\sinh^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(2^n n!)^2 (2n+1)} x^{2n+1}$ for $\lvert x\rvert \le 1$ $\sinh^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\ (2n-1)!!}{(2n+1)(2n)!!} x^{2n+1}$	$x - \frac{x^3}{2\cdot 3} + \frac{1\cdot 3x^5}{2\cdot 4\cdot 5} - \frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \frac{1\cdot 3\cdot 5\cdot 7x^9}{2\cdot 4\cdot 6\cdot 8\cdot 9} - \cdots$
$\cosh^{-1}(x) = \frac{\pi}{2} i - i \sum_{n=0}^{\infty} \frac{2^{-n}}{n!\,(2n+1)} x^{2n+1}$ for $\lvert x\rvert \le 1$	$\frac{\pi\, i}{2} - i\, x - \frac{i\, x^3}{2\cdot 3} - \frac{i\cdot 1\cdot 3x^5}{2\cdot 4\cdot 5} - \frac{i\cdot 1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} - \cdots$
$\tanh^{-1}(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$ for $\lvert x\rvert < 1, x \ne \pm 1$	$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \frac{x^{15}}{15} + \cdots$
$\operatorname{sech}^{-1}(x) = -\ln(x) + \ln(2) + \frac{1}{4} \sum_{n=0}^{\infty} \frac{(2n+1)!\ x^{2n+2}}{4^n [(n+1)!]^2}$	$-\ln(x) + \ln(2) + \frac{1}{4}x^2 + \frac{3i}{32}x^4 + \frac{5i}{96}x^6 + \cdots$
$\operatorname{csch}^{-1}(x) = -\ln(x) + \ln(2) + \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n\ (2n+1)!\ x^{2n+2}}{4^n [(n+1)!]^2}$	$-\ln(x) + \ln(2) + \frac{1}{4}x^2 - \frac{3i}{32}x^4 + \frac{5i}{96}x^6 - \cdots$
$\coth^{-1}(x) = -\frac{i\pi}{2} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$	$= -\frac{i\pi}{2} + x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \cdots$

Constant	Series Summation	Expanded
1	**Telescoping** $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$	$\left(\frac{1}{1\cdot2}\right)+\left(\frac{1}{2\cdot3}\right)+\left(\frac{1}{3\cdot4}\right)+\left(\frac{1}{4\cdot5}\right)+\left(\frac{1}{5\cdot6}\right)+\left(\frac{1}{6\cdot7}\right)+\cdots$
2	**Geometric** $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$	$\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\frac{1}{512}+\cdots$
$\frac{1}{3}$	**Archimède** $\sum_{n=1}^{\infty} \frac{1}{4^n}$	$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\frac{1}{1{,}024}+\frac{1}{4{,}096}+\frac{1}{16{,}384}+\cdots$
π	**Nilakantha (15th Century) II** $3+\sum_{n=2}^{\infty} \frac{(-1)^n 4}{((2n-1)^3-(2n-1))}$	$3+\left(\frac{4}{3^3-3}\right)-\left(\frac{4}{5^3-5}\right)+\left(\frac{4}{7^3-7}\right)-\left(\frac{4}{9^3-9}\right)+\cdots$
	$3+\sum_{n=2}^{\infty} \frac{(-1)^n(4)}{(n)(n+1)(n+2))}$	$3+\left(\frac{4}{2\cdot3\cdot4}\right)-\left(\frac{4}{3\cdot4\cdot5}\right)+\left(\frac{4}{4\cdot5\cdot6}\right)-\left(\frac{4}{5\cdot6\cdot7}\right)+\cdots$
$\frac{1}{\pi}$	**Ramanujan-Sato** $\frac{2\sqrt{2}}{99^2}\sum_{n=0}^{\infty} \frac{(4n)!}{n!^4}\frac{26{,}390n+1103}{396^{4n}}$	$\frac{2\sqrt{2}}{99^2}\left(1103+\frac{659{,}832}{396^4}+\frac{135{,}785{,}160}{396^8}+\cdots\right)$
$\frac{\pi}{4}$	**James Gregory (or Leibniz)** $\sum_{n=1}^{\infty} \frac{(-1)^{i+1}}{(2n-1)}$	$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}+\cdots$
	$\sum_{n=0}^{\infty} \frac{\sin[(2n+1)\theta]}{(2n+1)}$ $where\ 0<\theta<\pi$	$\frac{\sin(\theta)}{1}+\frac{\sin(3\theta)}{3}+\frac{\sin(5\theta)}{5}+\frac{\sin(7\theta)}{7}+\cdots$
$\frac{\pi}{16}$	**Nilakantha (15th Century) I** $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^5+4\cdot(2n-1)}$	$\left(\frac{1}{1^5+4\cdot1}\right)+\left(\frac{1}{3^5+4\cdot3}\right)+\left(\frac{1}{5^5+4\cdot5}\right)+\cdots$
$\frac{\pi^2}{6}$	**Basel Problem with Zeta $\boldsymbol{\zeta(2)}$** (solved by Leonhard Euler) $\sum_{n=1}^{\infty} \frac{1}{n^2}$	$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9^2}+\cdots$
$\frac{\pi^2}{8}$	**Leonhard Euler** $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$	$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{15^2}+\frac{1}{17^2}+\cdots$
$\frac{\pi^2}{12}$	**Euler's Alternating** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$	$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\frac{1}{7^2}-\frac{1}{8^2}+\frac{1}{9^2}-\frac{1}{10^2}+\cdots$

$\frac{\pi^2}{32}$	**Euler's Alternating** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$	$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{15^3} + \frac{1}{17^3} - \cdots$
$\frac{\pi^4}{90}$	**Zeta $\zeta(4)$** $\sum_{n=1}^{\infty} \frac{1}{n^4}$	$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} + \frac{1}{9^4} + \cdots$
$\frac{\pi^6}{945}$	**Zeta $\zeta(6)$** $\sum_{n=1}^{\infty} \frac{1}{n^6}$	$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \frac{1}{7^6} + \frac{1}{8^6} + \frac{1}{9^6} + \cdots$
e	$\sum_{n=0}^{\infty} \frac{1}{n!}$	$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \cdots$
$\frac{1}{e}$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$	$\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \cdots$
$ln(2)$	**Alternating Harmonic** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$	$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots$
	$\sum_{n=1}^{\infty} \frac{1}{2^n n}$	$\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} + \cdots$
	$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n)}$	$\left(\frac{1}{1 \cdot 2}\right) + \left(\frac{1}{3 \cdot 4}\right) + \left(\frac{1}{5 \cdot 6}\right) + \left(\frac{1}{7 \cdot 8}\right) + \left(\frac{1}{9 \cdot 10}\right) + \cdots$

Bernoulli Numbers	Euler Numbers	Gamma Function
$B_0 = 1$ $B_1 = -\frac{1}{2}$ $B_2 = \frac{1}{6}$ $B_4 = -\frac{1}{30}$ $B_6 = \frac{1}{42}$ $B_8 = -\frac{1}{30}$ $B_{10} = \frac{5}{66}$ $B_{12} = -\frac{691}{2{,}730}$ $B_{14} = \frac{7}{6}$ $B_{16} = -\frac{3{,}617}{510}$ $B_{18} = \frac{438{,}675}{798}$ $B_{20} = -\frac{174{,}611}{330}$ $B_{22} = \frac{854{,}513}{138}$	$E_0 = 1$ $E_1 = 0$ $E_2 = -1$ $E_3 = 0$ $E_4 = 5$ $E_5 = 0$ $E_6 = -61$ $E_8 = 1{,}385$ $E_{10} = -50{,}521$ $E_{12} = 2{,}702{,}765$ $E_{14} = -199{,}360{,}981$ $E_{16} = 19{,}391{,}512{,}145$ $E_{18} = -2{,}404{,}879{,}675{,}441$ $E_{20} = 370{,}371{,}188{,}237{,}525$ $E_{22} = -69{,}348{,}874{,}393{,}137{,}901$ $E_{2n+1} = 0$	$\Gamma_0 = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ $\Gamma_1 = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ $\Gamma_2 = \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$ $\Gamma_3 = \Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8}$ $\Gamma_4 = \Gamma\left(\frac{9}{2}\right) = \frac{105\sqrt{\pi}}{16},$ $\Gamma_5 = \Gamma\left(\frac{11}{2}\right) = \frac{945\sqrt{\pi}}{32}$ $\Gamma_6 = \Gamma\left(\frac{13}{2}\right) = \frac{10{,}395\sqrt{\pi}}{64}$ $\Gamma_7 = \Gamma\left(\frac{15}{2}\right) = \frac{135{,}135\sqrt{\pi}}{128}$ $\Gamma_8 = \Gamma\left(\frac{17}{2}\right) = \frac{2{,}027{,}025\sqrt{\pi}}{256}$ $\Gamma_9 = \Gamma\left(\frac{19}{2}\right) = \frac{34{,}459{,}425\sqrt{\pi}}{512}$ $\Gamma_{10} = \Gamma\left(\frac{21}{2}\right) = \frac{654{,}729{,}075\sqrt{\pi}}{1{,}024}$
Generating Function		
$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n$	$\frac{1}{\cosh(t)} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n$	$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x}\, dx$ $\Gamma\left(\frac{1}{2} + n\right) = \frac{(2n)!}{4^n\, n!}\sqrt{\pi}$ $\Gamma\left(\frac{n}{2}\right) = \frac{(n-2)!!}{2^{\frac{n-1}{2}}}\sqrt{\pi}$
Recursive Definition	**Iterated Sum**	**Recursive Definition**
$B_m(n)$ $= n^m - \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k(n)}{m-k+1}$ $B_0(n) = 1$	E_{2n} $= i \sum_{k=1}^{2n+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j (k-2j)^{2n+1}}{2^k\, i^k\, k}$	$\Gamma(n+1) = n \cdot \Gamma(n)$ $\Gamma\left(\frac{n}{2}\right) = \left(\frac{n-2}{2}\right) \cdot \Gamma\left(\frac{n-2}{2}\right)$ $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Sources

- https://www.wolframalpha.com
- https://en.wikipedia.org
- http://ddmf.msr-inria.inria.fr/1.9.1/ddmf
- http://web.mit.edu/kenta/www/three/taylor.html

See Also

- Harold's Taylor Series Cheat Sheet
- Harold's Infinite Products Cheat Sheet