**Harold’s Ordinary Differential Equations (ODE)**

**Cheat Sheet**

22 September 2025

**Classification**

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| **Term** | **Definition** |
| **By Type** |  |
| **Differential Equation (DiffEq)** | A mathematical equation that relates a function to its derivatives. |
| **Ordinary Differential Equation (ODE)** | A differential equation that involves an unknown function and its derivatives with respect to a single independent variable. For example: |
| **Partial Differential Equation (PDE)** | A type of equation that involves an unknown function and its partial derivatives with respect to multiple independent variables. For example: |
| **Systems of Differential Equations** | Uses vector notation. For example: |
| **By Order** |  |
| **Order** | The order of the highest derivative in the equation.  = 1st order  = 2nd order  = 3rd order  = 4th order |
| **By Linearity** |  |
| **General Solution** | A family of functions that has parameters and does not take any initial conditions into account. |
| **Particular Solution**  (Actual Solution) | Has no arbitrary parameters and satisfies the initial conditions. |
| **Singular Solution** | Cannot be obtained from the general solution. |
| **Linear Equation** |  |
| **Nonlinear Equation** | An equation that contains functions of such as , or functions of the derivatives of , such as . |

**Terms and Notation**

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| **Notation** | **Expanded Form** | **Description** |
|  |  | Variables for space and time |
|  | or | The function of a single independent variable. |
|  |  | The derivative of with respect to . |
|  |  | The partial derivative of with respect to . |
|  | Particular solution to the equation. | |
| **IVP** | Initial value problem. An initial condition that specifies the value of the unknown function at a given point in the domain so a particular solution can be found. e.g., | |
|  | Gamma Function | The Gamma function is the continuous version of the discrete factorial function, . |

**Direction Fields**

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| **Name** | **Slope Equation** | **Graph** |
| **Direction Field**  (Slope Field) | **General** Solution: |  |
| **Solution Curve** | Initial Condition:  **Particular** Solution: |  |

**1st-Order Linear**

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| **Technique** | **Form of Equation** | **Solution Method** |
| **1st-Order** |  |  |
| **Single Variable** |  |  |
| **Separable Equations**  (Separation of Variables) |  |  |
| **Integration Factor**  **( = )** |  | 1. Multiplying the entire equation by . 2. But the derivative of the product is: |
| **Exact Equations** |  | Find by integrating and comparing: |
| **Non-Exact Equations** | Reduction to Exact via Integrating Factor. | |
| **If** | **Then** |
| **Case 1:** |  |  |
| **Case 2:** |  |  |
| **Case 3:** |  |  |
| **Bernoulli** |  | If or , this is linear, else  1.Divide by  2. Change variable  3. Apply substitution to get a linear form  4. Use an integrating factor to solve |
| **Homogeneous** | where and are homogeneous functions of the same degree n, meaning: | 1. Start with:  2. Reduce to separation of variables using:  3. Product rule:   1. Rewrite in terms of . |
| **Substitution** |  | 1. Set  2. Solve for  3. Find in terms of  4. Rewrite the equation in terms of |
| **Reduction by Translation** |  | |
| **If** | **Then** |
| **Case 1: Intersecting Lines** |  | 1. Put  2. Find and  3. Solve using separation of variables  4. Translate back |
| **Case 2: Parallel Lines** |  | 1. Put  2. Solve |

**2nd-Order Linear Homogeneous ()**

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| **Technique** | **Form of Equation** | **Solution Method** |
| **2nd-Order** |  | Typical case has all constant coefficients instead of functions of x. |
| **Homogeneous** |  |  |
| **Principle of Superposition** | If  has solution  and  has solution | then  has solution |
| **Characteristic Equation** (Constant Coefficients) |  | All solutions are in the form:  Roots: |
| **If** | **Then** |
| **Case 1: Distinct Roots** |  | 1. Set  2. Superposition gives |
| **Case 2: Repeated Root** |  |  |
| **Case 3: Complex Roots** | with | 1. Set  2. Superposition gives  3. Apply  4. |
| **Reduction of Order** |  | If we already know ,  1. Put  2. Expand in terms of  3. Put  4. Solve the reduced equation |
| **Wronskian**  (Linear Independence) | and are linear independent iff |  |
| **Euler-Cauchy Equation** | where | 1. of the form  2. Reduction to Constant Coefficients:  Use  3. Rewrite in terms of using chain rule. |
| **Case 1: Distinct Roots** |  |  |
| **Case 2: Repeated Root** |  |  |
| **Case 3: Complex Roots** |  |  |

**2nd-Order Linear Non-Homogeneous ()**

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| **Technique** | **Form of Equation** | **Solution Method** |
| **2nd-Order** |  | where the general solution includes the particular solution, . |
| **Non-Homogeneous** |  | If homogeneous, then . |
| **Case 1: Missing** |  |  |
| **Case 2: Missing** |  | 1. Change of variable:  2. Solve twice |
| **Case 3: Missing** |  | 1. Change of variable: chain rule  2. is a 1st-order ODE  3. Solve it  4. Back-replace  5. Solve again |
| **Case 4: Missing** |  | 1. Change of variable: chain rule  2. is a separation of variables  3. Solve it  4. Back-replace  5. Solve again |
| **Method of Undetermined Coefficients**  (Guesswork) | Assume has the same form as with undetermined constant coefficients.  Valid forms:  Failure case: If any term of is a solution of , multiply by and repeat until it works. | |

**Laplace Transforms**

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| **Technique** | **Equation** | | **Solution** |
| **Laplace Transform** | * An integral transform that converts a function of a real variable (usually , in the time domain) to a function of a complex variable in the complex-valued frequency domain ( domain).   + NOTE: Frequency * Transforms ODEs into algebraic equations. * The substitution makes the integral look like a formal power series, with being the coefficient of . * The z-transform is the discrete analog of the Laplace transform in signal processing. | | |
| **Inverse Laplace Transform** |  | | |
| **Laplace Transform Equation** | where   * must be defined on * is an arbitrary real variable * The improper integral must converge | | |
| **Transform of a Derivative** | **Theorem**: If are continuous on [0, and are of exponential order, and if is piecewise continuous on [0,, then the Laplace transform of the nth derivative of is given by the general nth order derivative equation below. | | |
| **Laplace Transform of Derivatives** | First-order derivative |  | |
| Second-order derivative |  | |
| Third-order derivative |  | |
| nth order derivative |  | |
| **Using Laplace to Solve a DiffEq** | Solves IVPs for linear differential equations with constant coefficients.   1. Apply the Laplace transform to each term in the IVP. 2. Solve the resulting algebraic equation for the transformed function . 3. Apply the inverse Laplace transform to find the solution. | | |

**Table of Laplace Transforms ()**

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| 41 | [Heaviside Function](https://tutorial.math.lamar.edu/classes/de/StepFunctions.aspx) |  |
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| 43 | [Dirac Delta Function](https://tutorial.math.lamar.edu/classes/de/DiracDeltaFunction.aspx) |  |
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**Fourier Transforms**

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| **Technique** | **Equation** | **Solution** |
| **Fourier Series** | * A method to **decompose** a periodic signal into a sum of sine and cosine functions with different frequencies and amplitudes. * Is particularly useful for analyzing arbitrary **periodic** signals. * Was introduced by Joseph Fourier. * Is widely used in fields such as electronics, signal processing, and quantum mechanics. | |
| **Fourier Transform** | * A mathematical operation that **transforms** a signal from the time domain () to the frequency domain (). * Unlike the Fourier Series, the Fourier Transform can be applied to both **periodic** and **non-periodic** signals. * Are widely used to solve differential equations.   fourier pair, and fourier transforms | |
| **Fourier Transform Equation** | where   * must be defined on * is an arbitrary real variable where * The improper integral must converge | |
| **Inverse Fourier Transform Equation** | where   * must be defined on * is an arbitrary real variable where * The improper integral must converge | |
| **Example** |  | |
| **Using Fourier Transforms to Solve a DiffEq** | Solves ODEs for linear differential equations with constant coefficients.   1. Apply the Fourier transform to each term in the ODE. 2. Solve the resulting algebraic equation for the transformed function . 3. Apply the inverse Fourier transform to find the solution. | |

**Table of Fourier Transforms ()**

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**Sources**

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* Furius Enterprises (2010). Differential Equations Cheat sheet 2nd-order Homogeneous. <https://furius.ca/cqfpub/doc/diffequs/diffequs.pdf>
* Shapiro, B.E. (2014). Table of Laplace Transforms. <https://www.integral-table.com/downloads/LaplaceTable.pdf>
* Wikipedia (2025). Tables of important Fourier transforms. <https://en.wikipedia.org/wiki/Fourier_transform#Tables_of_important_Fourier_transforms>

**See Also**

* + [Harold’s Partial Differential Equations (PDE) Cheat Sheet](https://www.toomey.org/tutor/harolds_cheat_sheets/Harolds_PDE_Cheat_Sheet.pdf)
  + [Harold's Differential Equation Models Cheat Sheet](https://www.toomey.org/tutor/harolds_cheat_sheets/Harolds_DiffEq_Models_Cheat_Sheet.pdf)
  + [Harold's Euler's Method Example](https://www.toomey.org/tutor/harolds_cheat_sheets/Harolds_DiffEq_Eulers_Method_Cheat_Sheet.pdf)