**Harold’s Counting**

**Cheat Sheet**

12 February 2025

(See also Harold’s Sets Cheat Sheet)

**Counting Rules**

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| **Rule** | **Description** | **Comments** |
| **Cardinality** | The cardinality of A is the number of elements in set A = $|A|$ | * if A = {(1,2), (3,4), (5,6)}, then |A| = 3
* Also denoted $n(A)$
* Cardinality = Counting
 |
| **Product Rule** | Let $A\_{1}, A\_{2}, ..., A\_{n}$ be finite sets. Then,$$|A\_{1}×A\_{2}× ⋅⋅⋅ ×A\_{n}|=|A\_{1}| ⦁ |A\_{2}| ⦁⋅⋅⋅⦁ |A\_{n}|$$ | * Counts sequences
* Think Intersection (∩)
 |
| **Sum Rule** | Consider n sets, $A\_{1}, A\_{2}, ..., A\_{n}$. If the sets are mutually disjoint (Ai ∩ Aj = ∅ for i ≠ j), then$$|A\_{1}∪A\_{2}∪… ∪A\_{n}| =|A\_{1}|+|A\_{2}|+… +|A\_{n}|$$ | * Counts sequences
* Think Union (∪)
 |
| **Generalized Product Rule** | $$\left|S\right|=n\_{1}⦁n\_{2}⦁⋅⋅⋅⦁n\_{k}$$*n! = (n)(n-1)(n-2) … (2)(1)* | * In selecting an item from a set, if the number of choices at each step is independent, then the number of items in the set is the product of the number of choices in each step.
 |
| **Bijection Rule** | Let S and T be two finite sets.If there is a bijection from S to T, then$$\left|S\right|=\left|T\right|$$ | * 1-to-1 Correspondence
 |
| **k-to-1 Rule** | $$|Y|=\frac{|X|}{k}$$ | * k-to-1 Correspondence
 |

**Counting Formulas & Techniques**

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| **Rule** | **Description** | **Comments** |
| **Factorial** | $$n!=n ⦁ \left(n-1\right) ⦁ \left(n-2\right) ⦁…⦁ 3 ⦁ 2 ⦁ 1$$ | * The number of permutations of a finite set with n elements is P(n,n)
 |
| **Permutation** | $$P\left(n,r\right)=\_{r}=\frac{n!}{\left(n-r\right)!}$$$$ =n(n-1) ... (n-r+1)$$ | * Order matters ( , )
* r-permutation
* Counting sequences
* Common application of the generalized product rule
* Order can be fixed but arbitrary
* Elements cannot be repeated
* Use when elements are all different
 |
| **Grouping** | Password length is 18.No character repeats.Must contain: a, z, 1, and 9.$$P(18, 4) ⦁ P(36-4, 18-4)$$ | * Combines permutation with product rule
 |
| **Combination** | $$C\left(n,r\right)=\_{r}=\left(\genfrac{}{}{0pt}{}{n}{r}\right)$$$$=\frac{n!}{r!\left(n-r\right)!}$$ | * Order does not matter { , }
* Counting subsets
* r-combination
* n choose r
* Counting the r-subsets
* Combination = subset
* Use when elements are all identical
 |
| Identity:$$\left(\genfrac{}{}{0pt}{}{n}{n-r}\right)=\left(\genfrac{}{}{0pt}{}{n}{r}\right)$$ | * An equation is called an **identity** if the equation holds for all values for which the expressions in the equation are well defined.
 |
| **Counting Subsets** | Bijection from:5-bit strings with exactly 2 1's To:2-subsets of { 1, 2, 3, 4, 5 } =$$\left(\genfrac{}{}{0pt}{}{5}{2}\right)=10$$ | * Binary example
* Counting Strings by Counting Subsets
 |
| $$\left(\genfrac{}{}{0pt}{}{m+n}{m}\right)$$ | * Counting paths on a grid {N, E}
* Binary example if m = #1s & n = #0s
 |

**Counting with Discrete Probability**

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| --- | --- | --- |
| **Rule** | **Formula** | **Definition** |
| **Notation** | $∩$ = Intersection or “”and”$∪$ = Union or “or”$\overbar{˽}= $Negation or “not” | * “and” implies multiplication.
* “or” implies addition.
* “not” implies negation.
 |
| **Independent** | If $P\left(B\right)=P(A)$ | * The occurrence of one event does not affect the probability of the other event, or vice versa.
 |
| **Mutually Independent** | No sets overlap.Future outcomes are not impacted by previous outcomes. | * Applies to more than two events
 |
| **Dependent** | If $\left|A∩B\right|\ne 0$ | * The occurrence of one event affects the probability of the other event.
 |
| **Disjoint**(“mutually exclusive”) | If $\left|A∩B\right|=0$, then$$\left|A∩B\right|=\left|A\right|+\left|B\right|$$ | * The events can never occur together.
 |
| **Probability**(“likelihood”) | $$P\left(E\right)= \frac{\left|E\right|}{\left|S\right|}$$ | * *S* = **S**ample space or entire set
* *A, B, E* = **E**vent or subset

$$0\leq P\left(E\right)\leq 1$$ |
| **Addition Rule** (“or”) | $$\left|A∪B\right|$$$$=|A|+|B|-|A∩B|$$ | **Inclusion-Exclusion Principle*** Let A, B and C be three finite sets, then …
* If sets overlap, then don’t double count
* ”… in any of the 3.”
* ”… divisible by 2, 3,or 5.”
 |
| $$\left|A∪B∪C\right|$$$$=\left|A\right|+\left|B\right|+\left|C\right|-\left|A∩B\right|-\left|B∩C\right|-\left|A∩C\right|+|A∩B∩C|$$ |
| $$\left|A∪B∪C∪D\right|$$$$=\left|A\right|+\left|B\right|+\left|C\right|+\left|D\right|$$$$- \left|A∩B\right|-\left|A∩C\right|-\left|A∩D\right|-\left|B∩C\right|-\left|B∩D\right|-\left|C∩D\right|$$$$+ \left|A∩B∩C\right|+\left|A∩B∩D\right|+\left|A∩C∩D\right|+\left|B∩C∩D\right|$$$$- |A∩B∩C∩D|$$ |
| if mutually independent / disjoint:$$\left|A\_{1}∪A\_{2}∪…∪A\_{n}\right|=$$$$|A\_{1}|+|A\_{2}|+…+|A\_{n}|$$ | * A collection of sets is **mutually disjoint** if the intersection of every pair of sets in the collection is empty.
* Restatement of the Sum Rule
 |
| **Multiplication Rule**(“and”) | $$\left|A∩B\right|=\left|A\right|·\left|\left( A\right)\right|$$$$\left|A∩B\right|=\left|B\right|·\left|\left( B\right)\right|$$$$\left|A∩B\right|=\left|A\right|-\left|A∩\overline{B}\right|$$if independent / disjoint:$$\left|A∩B\right|=\left|A\right|·\left|B\right|$$if mutually independent / disjoint:$$\left|A∩B∩C\right|=\left|A\right|·\left|B\right|·\left|C\right|$$$$\left|A\_{1}∩A\_{2}∩ ... ∩A\_{n}\right|=\left|A\_{1}\right|·\left|A\_{2}\right|·...·\left|A\_{n}\right|$$ |  |
| **Complement Rule / Subtraction Rule**(“not”) | $$P\left(S\right)=P\left(E∪\overline{E}\right)$$$$\left|E\right|+\left|\overline{E}\right|=\left|S\right|$$$$|E|=|S|-|\overbar{E}|$$$$|\overbar{E}|=|S|-|E|$$$$\left|\left( B\right)\right|+\left|\left( B\right)\right|=\left|A\right|$$ | * S = entire set, E = subset
* The complement of event *E* (denoted $\overline{E} or E^{c})$ means “**not *E***”;
* It consists of all simple outcomes that are not in *E*.
* ”has at least one” so choose $\overbar{E} $as “none”
 |
| **Union by Compliment** | $$\left|S\right|-\left|\overbar{E\_{1}∪E\_{2}∪...∪E\_{n}}\right|=$$$$|E\_{1}∪E\_{2}∪...∪E\_{n}|$$ | * S = U = Universal set (all)
* E.g., 104 - 94
 |
| **Conditional Probability**(“given that”) | $$P\left( B\right)=\frac{\left|A∩B\right|}{\left|B\right|}$$if independent / disjoint:$$P\left( B\right)=\frac{P(A∩B)}{P(B)}=P(A)$$$$\left|\left( B\right)\right|=\left|A\right|$$$$\left|\left( A\right)\right|=\left|B\right|$$ | * Means the probability of event A given that event B has already occurred.
* Is a rephrasing of the Multiplication Rule.
* P(A|B) is the proportion of elements in B that are ALSO in A.
 |
| **Total Probability Rule** | $$P\left(A\right)=P(A∩B\_{1})+…+P(A∩B\_{n})$$$$=P\left(B\_{1}\right)·P\left( B\_{1}\right)+…+P\left(B\_{n}\right)·P\left( B\_{n}\right)$$$$P\left(A\right)=P(A∩B)+P(A∩\overbar{B})$$$$=P\left( B\right)·P\left(B\right)+P\left( \overbar{B}\right)·P\left(\overbar{B}\right)$$ | * To find the probability of event A, partition the sample space into several disjoint events.
* A must occur along with one and only one of the disjoint events.
 |
| **Bayes’ Theorem** | $$P\left( B\right)=\frac{\left|A∩B\right|}{\left|B\right|}=\frac{\left|\left( A\right)\right|·\left|A\right|}{\left|B\right|}$$$$=\frac{\left|\left( A\right)\right|·\left|A\right|}{\left|\left( A\right)\right|·\left|A\right|+\left|\left( \overbar{A}\right)\right|·\left|\overbar{A}\right|}$$ | * Allows P(A|B) to be calculated from P(B|A).
* Meaning it allows us to reverse the order of a conditional probability statement, and is the only generally valid method!
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**Sources**:

* [SNHU MAT 230](https://www.snhu.edu/admission/academic-catalogs/coce-catalog#/courses/4kVhSZLtg) - Discrete Mathematics, zyBooks.