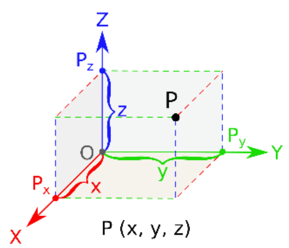
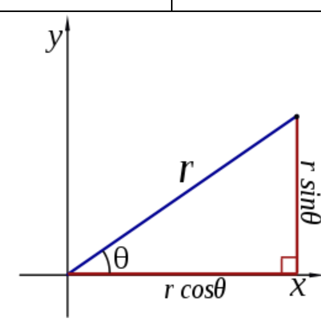
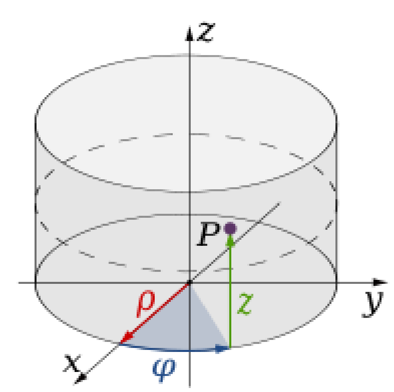
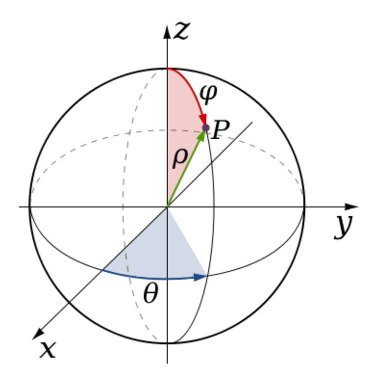
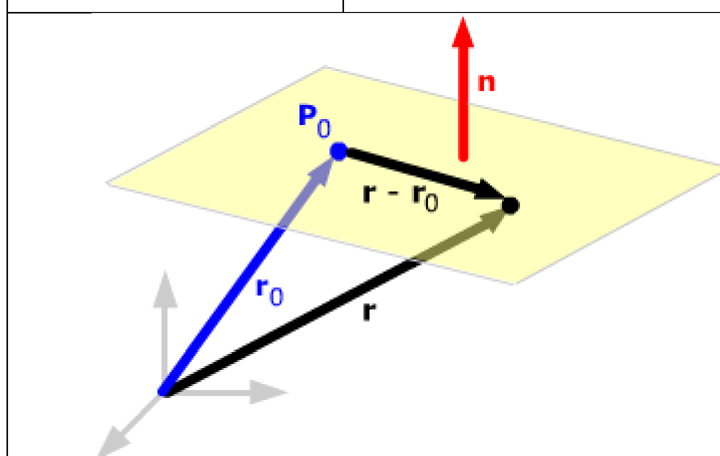
Harold's Calculus 3
in Multiple Coordinate Systems
Cheat Sheet

1 September 2025

	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Point	2D: $f(x) = y$ (x, y) or (a, b) 3D: $f(x, y) = z$ (x, y, z) 4D: $f(x, y, z) = w$ (x, y, z, w) 	(r, θ) or $r \angle \theta$ *Polar → Rect.* $x = r \cos \theta$ $y = r \sin \theta$ $z = z$ $\tan \theta = \frac{y}{x}$ *Rect. → Polar* $r^2 = x^2 + y^2$ $r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$	(ρ, θ, ϕ) $x = \rho \sin \phi \cos \theta$ $y = \rho \sin \phi \sin \theta$ $z = \rho \cos \phi$ $\rho^2 = r^2 + z^2$ $\rho^2 = x^2 + y^2 + z^2$ $\tan \theta = \left(\frac{y}{x}\right)$ $\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$ $\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$	*Point (a, b) in Rectangular :* $x(t) = a$ $y(t) = b$ $< a, b >$ $t = 3^{rd}$, *variable, usually time, with 1 degree of freedom (df)*	$\vec{r} = \langle x_0, y_0, z_0 \rangle$ $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$	$[a] \quad [x] = [b]$
Line	*Slope-Intercept Form:* $y = mx + b$ *Point-Slope Form:* $y - y_0 = m(x - x_0)$ *Normal Form:* $Ax + By + C = 0$ *where A and B* $\neq 0$ *Calculus Form:* $f(x) = f'(a)\,x + f(0)$ *where* $m = f'(a)$ *Intercept Form:* $\frac{x}{a} + \frac{y}{b} = 1$ 3D: $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$			$< x, y > = < x_0, y_0 > + t < a, b >$ $< x, y > = < x_0 + at, y_0 + bt >$ *where* $< a, b > = < x_2 - x_1, y_2 - y_1 >$ $x(t) = x_0 + ta$ $y(t) = y_0 + tb$ $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{b}{a}$	$\vec{r} = \vec{r_0} + t\,\vec{v}$ $= \langle x_0, y_0, z_0 \rangle$ $+ t \langle a, b, c \rangle$ 	$[a \quad b] \begin{bmatrix} x \\ y \end{bmatrix} = [c]$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$

	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Plane	*Dot Product of Point-Normal Form:* $n_x(x-x_0)+n_y(y-y_0)+n_z(z-z_0)=0$ where: $\boldsymbol{n}=\langle n_x,n_y,n_z\rangle=\langle a,b,c\rangle$ *is the normal vector* *General Form:* $Ax+By+Cz+D=0$ *Intercept Form:* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$	(r,θ,k) $(0\le r<\infty)$ $(0\le\theta<2\pi)$ *where* r *and* θ *take on all values in their domains*	(ρ,θ,k) $(0\le\rho<\infty)$ $(0\le\theta<2\pi)$ *where* ρ *and* θ *take on all values in their domasins*	*Parametric Form:* $x=x_0+su_1+tv_1$ $y=y_0+su_2+tv_2$ $z=z_0+su_3+tv_3$ *where:* • (x_0,y_0,z_0) *is a point on the plane.* • $\langle u_1,u_2,u_3\rangle$ *and* $\langle v_1,v_2,v_3\rangle$ *are direction vectors on the plane.* • *s and t are parameters that vary over all real numbers.*	*Vector Form:* $\boldsymbol{r}=\boldsymbol{r}_0+s\boldsymbol{v}+t\boldsymbol{w}$ *where:* • $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *are given vectors defining the plane* • $\boldsymbol{r_0}$ *is the vector representing the position of an arbitrary (but fixed) point on the plane*	*Point-Normal Form:* $\boldsymbol{n}\bullet(\boldsymbol{r}-\boldsymbol{r}_0)=0$ *or* $n_x(x-x_0)+n_y(y-y_0)+n_z(z-z_0)=0$ Normal Vector: $\boldsymbol{n}=\boldsymbol{v_1}\times\boldsymbol{v_2}=\langle n_x,n_y,n_z\rangle$ $\boldsymbol{n}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ x_3-x_1 & y_3-y_1 & z_3-z_1\end{vmatrix}$ *Points on the plane:* • $P_0=(x_0,y_0,z_0)\rightarrow\boldsymbol{r_0}=\langle P_0\rangle$ • $P_1=(x_1,y_1,z_1)$ • $P_2=(x_2,y_2,z_2)$ • $P_3=(x_3,y_3,z_3)$ *Vectors on the plane:* • $\boldsymbol{v_1}=\langle P_2-P_1\rangle=\langle x_2-x_1,y_2-y_1,z_2-z_1\rangle$ • $\boldsymbol{v_2}=\langle P_3-P_1\rangle=\langle x_3-x_1,y_3-y_1,z_3-z_1\rangle$



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<tr><th></th><th>Rectangular</th><th>Polar/Cylindrical</th><th>Spherical</th><th>Parametric</th><th>Vector</th><th>Matrix</th></tr>
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<td>Conics</td>
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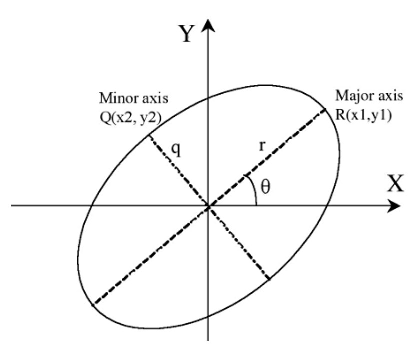
General Equation for All Conics:
$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
where
Line: $A = B = C = 0$
Circle: $A = C$ and $B = 0$
Ellipse: $AC > 0$
or $B^2 - 4AC < 0$
Parabola: $AC = 0$
or $B^2 - 4AC = 0$
Hyperbola: $AC < 0$
or $B^2 - 4AC > 0$
Note: If $A + C = 0$, then square hyperbola
Rotation:
If B ≠ 0, then rotate the coordinate system:
$\cot 2\theta = \frac{A - C}{B}$
$x = x' \cos\theta - y' \sin\theta$
$y = y' \cos\theta + x' \sin\theta$
New = (x′, y′), Old = (x, y) rotates through angle θ from x-axis


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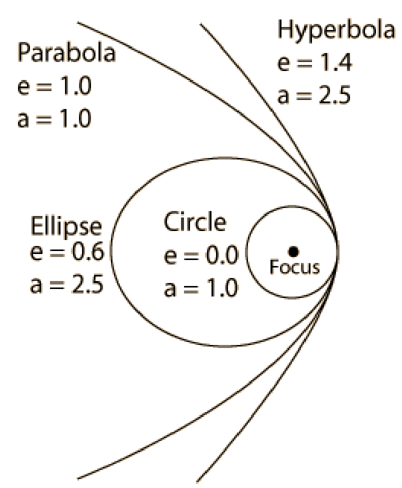
General Equation for All Conics:
Vertical Axis of Symmetry:
$r = \frac{p}{1 - e\cos\theta}$
Horizontal Axis of Symmetry:
$r = \frac{p}{1 - e\sin\theta}$
$\text{where } p = \begin{cases} a(1 - e^2) \\ 2d \\ a(e^2 - 1) \end{cases} \text{for} \begin{cases} 0 \le e < 1 \\ e = 1 \\ e > 1 \end{cases}$
p = semi-latus rectum or the line segment running from the focus to the curve in a direction parallel to the directrix
Eccentricity:
Circle $e = 0$
Ellipse $0 < e < 1$
Parabola $e = 1$
Hyperbola $e > 1$


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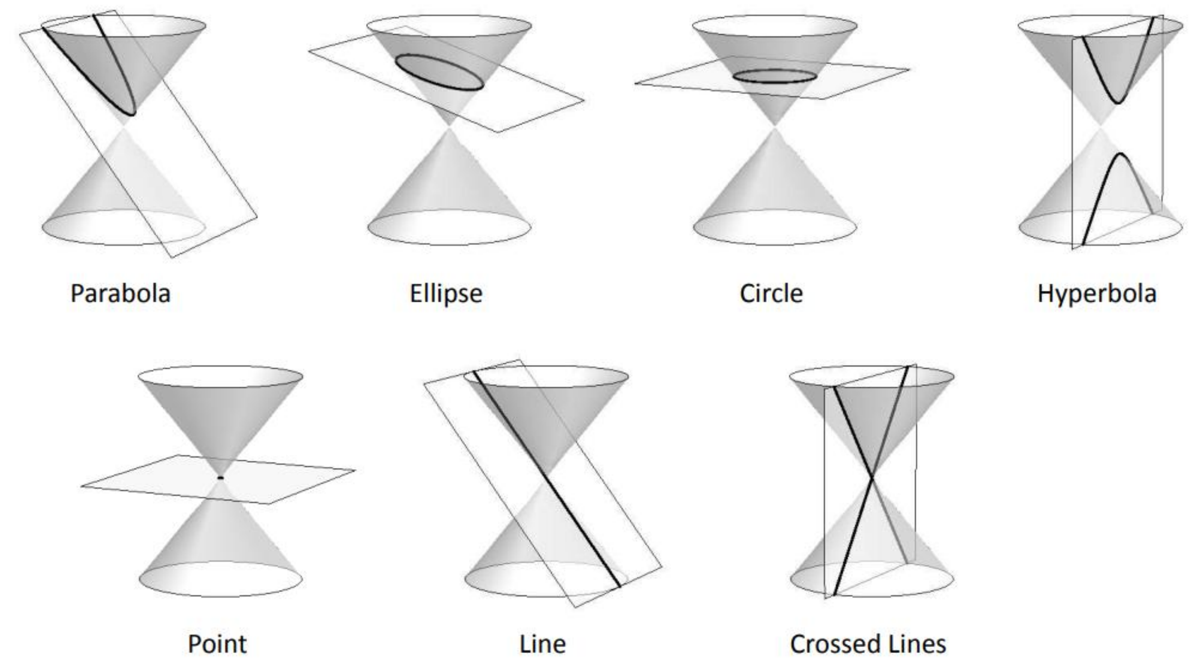

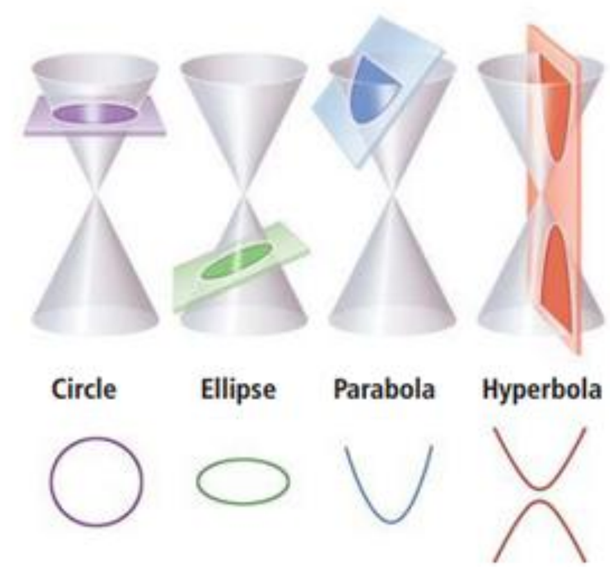

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Quadratic Form:
$(x \;\; y)\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (D \;\; E)\begin{pmatrix} x \\ y \end{pmatrix} + F = 0$
Matrix Form:
$\boldsymbol{x}^T A_Q \boldsymbol{x} = 0$
where
$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$
$A_Q = \begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix}$
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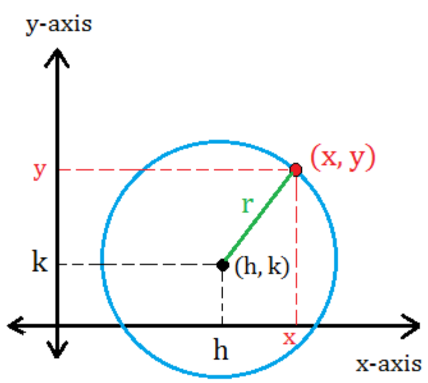
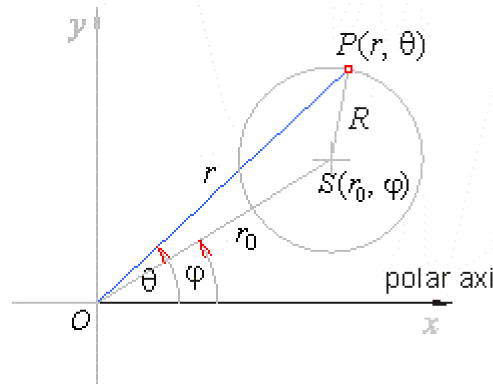
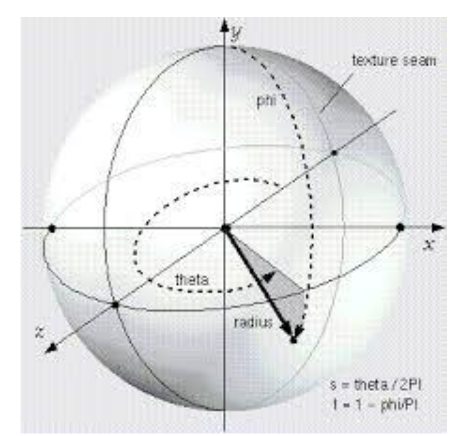
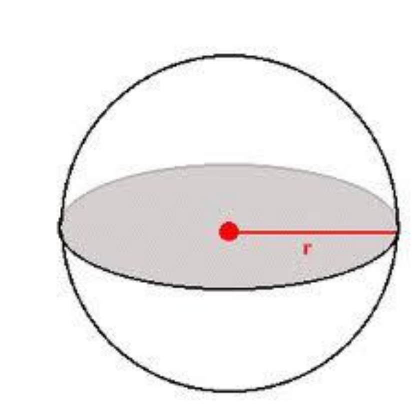
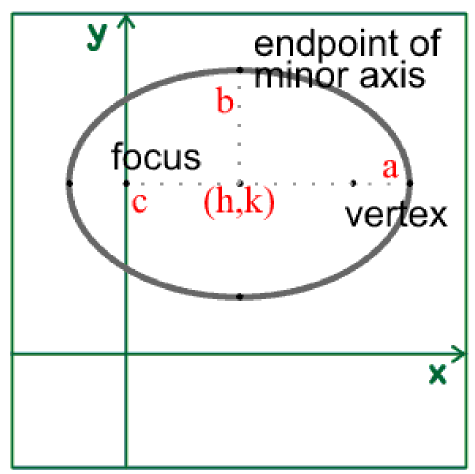
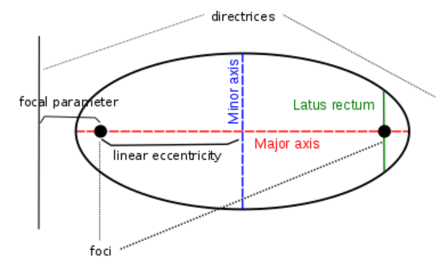
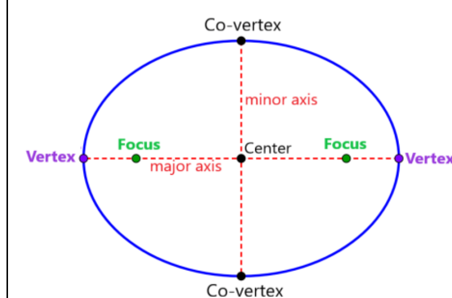
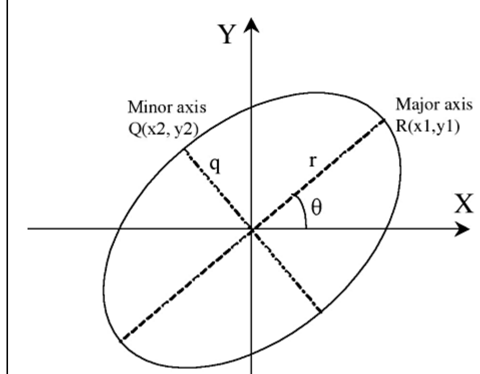
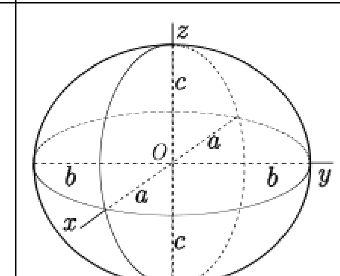
	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Circle	$x^2+y^2=r^2$ $(x-h)^2+(y-k)^2=r^2$ *General Form:* $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ $\text{where } A=C \text{ and } B=0$ *Center:* (h,k) *Vertices: NA* *Focus:* (h,k) 	*Centered at Origin:* *r = a (constant)* $\theta = \theta\ [0,2\pi]$ *or* $[0,360°]$ *Centered at* (r_0,ϕ)*:* $r^2+r_0^2-2rr_0\cos(\theta-\phi)=R^2$ *Hint: Law of Cosines* *or* $r=r_0\cos(\theta-\phi)+\sqrt{a^2-r_0^2\sin^2(\theta-\phi)}$ 	$\rho=\ constant$ $\theta=\ \theta\ [0,2\pi]$ $\phi=constant=0$	$x(t)=r\cos(t)+h$ $y(t)=r\sin(t)+k$ $[t_{min},\ t_{max}]=[0,2\pi]$ *Center:* (h,k) *Focus:* (h,k)	2D: $\lvert \boldsymbol{x}-\boldsymbol{m}\rvert=r$ **x** = vector of the points of the circle **m** = vector to the center of the circle r = radius 3D: $\boldsymbol{x}=\boldsymbol{c}+r\cos(t)\,\boldsymbol{u}+r\sin(t)\,\boldsymbol{v}$ $\boldsymbol{c}=(c_x,c_y,c_z)$ center of the circle $\boldsymbol{u},\boldsymbol{v}$ are two orthogonal unit vectors in the plane of the circle $t=[0,2\pi]$	$(x\ \ y)\begin{pmatrix}A & B/2\\ B/2 & C\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+(D\ \ E)\begin{pmatrix}x\\ y\end{pmatrix}+F=0$ *where* $\det\begin{pmatrix}A & B/2\\ B/2 & C\end{pmatrix}>0$ $A=C$ $B=0$
Sphere	$x^2+y^2+z^2=r^2$ $(x-h)^2+(y-k)^2+(z-l)^2=r^2$ *Focus and center:* (h,k,l) *General Form:* $Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0$ $\text{where } A=B=C>0$ *Cylindrical to Rectangular:* $x=r\cos(\theta)$ $y=r\sin(\theta)$ $z=z$ Spherical to Rectangular: $x=r\sin\theta\cos\phi$ $y=r\sin\theta\sin\phi$ $z=r\cos\theta$	*Rectangular to Cylindrical:* $r=\sqrt{x^2+y^2}$ *Spherical to Cylindrical:* $\rho=r\sin(\theta)$ $\phi=\phi$ $z=r\cos(\theta)$	$\rho=\ constant$ $\theta=\ \theta\ [0,2\pi]$ $\phi=\phi\ [0,2\pi]$ *Rectangular to Spherical:* $r=\sqrt{x^2+y^2+z^2}$ $\theta=\arccos\left(\frac{z}{r}\right)$ $\phi=\arctan\left(\frac{y}{x}\right)$ *Cylindrical to Spherical:* $r=\sqrt{\rho^2+z^2}$ $\theta=\arctan\left(\frac{\rho}{z}\right)=\arccos\left(\frac{z}{r}\right)$ $\phi=\phi$		Rectangular: $\boldsymbol{r}\equiv\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ Cylindrical: $\boldsymbol{r}\equiv\begin{bmatrix}r\cos(\theta)\\ r\sin(\theta)\\ z\end{bmatrix}$ Spherical: $\boldsymbol{r}\equiv\begin{bmatrix}r\sin\theta\cos\phi\\ r\sin\theta\sin\phi\\ r\cos\theta\end{bmatrix}$	

	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ *General Form:* $Ax^2+Bxy+Cy^2+Dx+Ey$ $+F=0$ *where* $B^2-4AC<0$ *or* $AC>0$ *Center:* (h,k) *Vertices:* $(h\pm a,k)$ *Co-Vertices:* $(h,k\pm b)$ *Foci:* $(h\pm c,k)$ *Focus length, c, from center:* $c^2 = a^2-b^2$ *Eccentricity:* $e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}$ *If B ≠ 0, then rotate coordinate system:* $\cot 2\theta = \frac{A-C}{B}$ $x = x'\cos\theta - y'\sin\theta$ $y = y'\cos\theta + x'\sin\theta$ *New = (x', y'), Old = (x, y) rotates through angle θ from x-axis*	*Vertical Axis of Symmetry:* $r=\frac{a\,(1-e^2)}{1\pm e\cos\theta}$ *Horizontal Axis of Symmetry:* $r=\frac{a\,(1-e^2)}{1\pm e\sin\theta}$ *Eccentricity:* $0<e<1$ $r(\theta)=\frac{a\,b}{\sqrt{(b\cos\theta)^2+(a\sin\theta)^2}}$ *relative to center* (h,k) 	  ***Interesting Note:*** *The sum of the distances from each focus to a point on the curve is constant.* $\lvert d_1+d_2\rvert = k$	$x(t)=a\cos(t)+h$ $y(t)=b\sin(t)+k$ $[t_{min},\,t_{max}]=[0,2\pi]$ *Center:* (h,k) *Rotated Ellipse:* $x(t)=a\cos t\cos\theta - b\sin t\sin\theta$ $+h$ $y(t)=a\cos t\sin\theta + b\sin t\cos\theta$ $+k$ *θ = the angle between the x-axis and the major axis of the ellipse* 	$\boldsymbol{x}$ $=\boldsymbol{c}+a\cos(t)\,\boldsymbol{u}$ $+b\sin(t)\,\boldsymbol{v}$ $\boldsymbol{c}=(c_x,c_y,c_z)$ center of the ellipse $\boldsymbol{u},\boldsymbol{v}$ are two orthogonal unit vectors in the plane of the ellipse $t=[0,2\pi s]$	$(x\ \ y)\begin{pmatrix}A & B/2\\ B/2 & C\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$ $+(D\ \ E)\begin{pmatrix}x\\ y\end{pmatrix}+F$ $=0$ *where* $\det\begin{pmatrix}A & B/2\\ B/2 & C\end{pmatrix}>0$
Ellipsoid	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}+\frac{(z-l)^2}{c^2}$ $=1$	$\frac{r^2\cos^2\theta}{a^2}+\frac{r^2\sin^2\theta}{b^2}+\frac{z^2}{c^2}=1$	$\frac{r^2\cos^2\theta\,\sin^2\phi}{a^2}$ $+\frac{r^2\sin^2\theta\,\sin^2\phi}{b^2}$ $+\frac{r^2\cos^2\phi}{c^2}=1$	$x(t,u)=a\cos(t)\cos(u)+h$ $y(t,u)=b\cos(t)\sin(u)+k$ $z(t,u)=c\sin(t)+l$ $[t_{min},\,t_{max}]=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ $[u_{min},\,u_{max}]=[-\pi,\pi]$ *Center:* (h,k,l)		$(\boldsymbol{x}-\boldsymbol{v})^T A^{-1}(\boldsymbol{x}-\boldsymbol{v})=1$ *Centered at vector* $\boldsymbol{v}$

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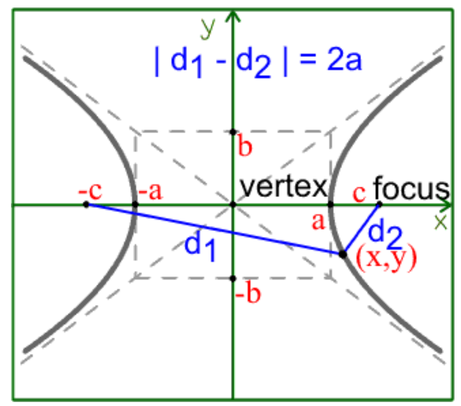
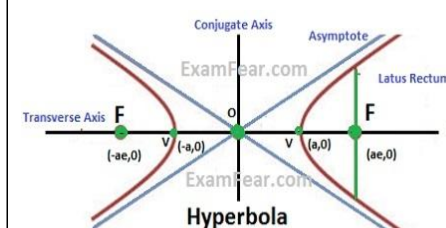
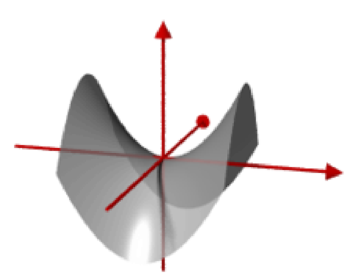
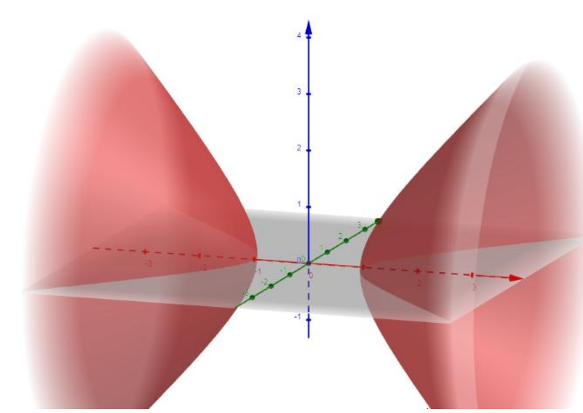
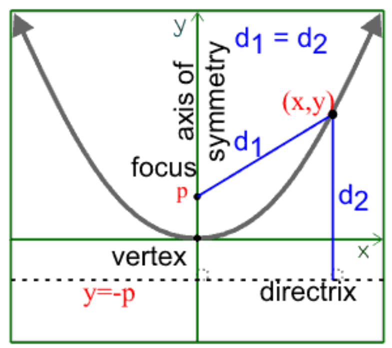
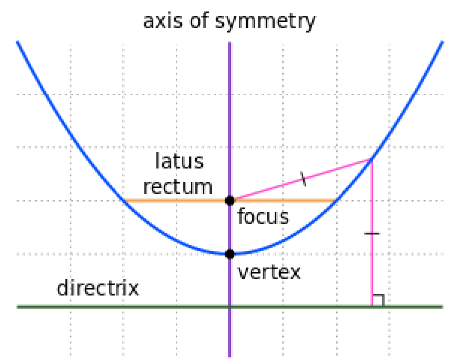
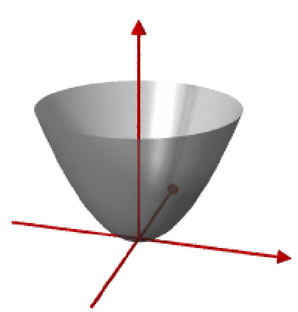
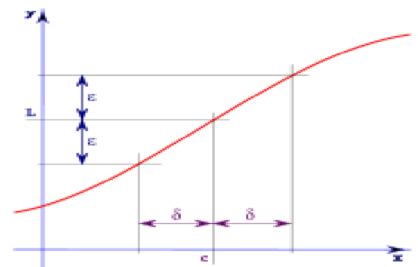
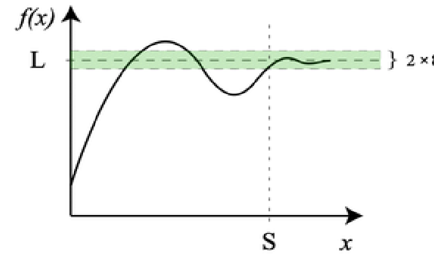
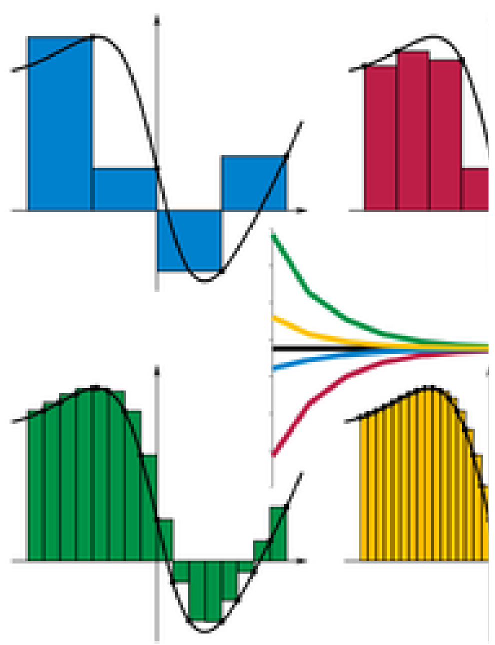
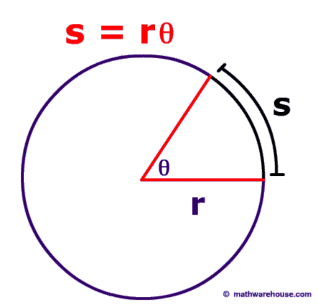
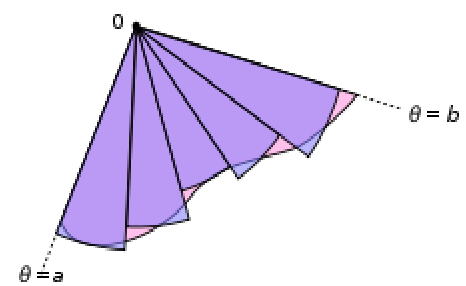
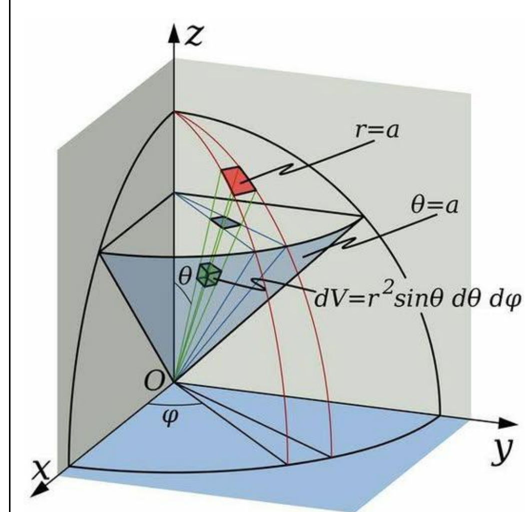
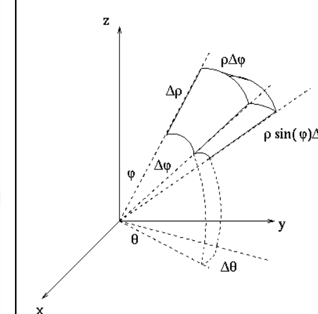
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ *General Form:* $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ *where* $B^2-4AC>0$ *or* $AC<0$ *If* $A+C=0$*, square hyperbola* *Center:* (h,k) *Vertices:* $(h\pm a,k)$ *Foci:* $(h\pm c,k)$ *Focus length, c, from center:* $c^2=a^2+b^2$ *Eccentricity:* $e=\frac{c}{a}=\frac{\sqrt{a^2+b^2}}{a}=\sec\theta$ *If B ≠ 0, then rotate coordinate system:* $\cot 2\theta=\frac{A-C}{B}$ $x=x'\cos\theta-y'\sin\theta$ $y=y'\cos\theta+x'\sin\theta$ *New = (x′, y′), Old = (x, y) rotates through angle θ from x-axis*	 ***Interesting Note:*** *The difference between the distances from each focus to a point on the curve is constant.* $\lvert d_1-d_2\rvert=k$	*Vertical Axis of Symmetry:* $r=\frac{a\,(e^2-1)}{1\pm e\cos\theta}$ *Horizontal Axis of Symmetry:* $r=\frac{a\,(e^2-1)}{1\pm e\sin\theta}$ *Eccentricity:* $e>1$ *where* $e=\frac{c}{a}=\frac{\sqrt{a^2+b^2}}{a}=\sec\theta>1$ *relative to center* (h,k) $-\cos^{-1}\left(-\frac{1}{e}\right)<\theta<\cos^{-1}\left(-\frac{1}{e}\right)$  *p = semi-latus rectum or the line segment running from the focus to the curve in the directions* $\theta=\pm\frac{\pi}{2}$	*Left-Right Opening Hyperbola:* $x(t)=a\sec(t)+h$ $y(t)=b\tan(t)+k$ $[t_{min},\,t_{max}]=[-c,c]$ *Vertex: (h, k)* *Alternate Form:* $x(t)=\pm a\cosh(t)+h$ $y(t)=b\sinh(t)+k$ *Up-Down Opening Hyperbola:* $x(t)=a\tan(t)+h$ $y(t)=b\sec(t)+k$ $[t_{min},\,t_{max}]=[-c,c]$ *Vertex: (h, k)* *Alternate Form:* $x(t)=a\sinh(t)+h$ $y(t)=\pm b\cosh(t)+k$ *General Form:* $x(t)=At^2+Bt+C$ $y(t)=Dt^2+Et+F$ *where A and D have different signs*		$\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}A & B/2\\ B/2 & C\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}D & E\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+F=0$ *where* $\det\begin{pmatrix}A & B/2\\ B/2 & C\end{pmatrix}<0$
Hyperboloid	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}-\frac{(z-l)^2}{c^2}=1$ $-\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}+\frac{(z-l)^2}{c^2}=1$					

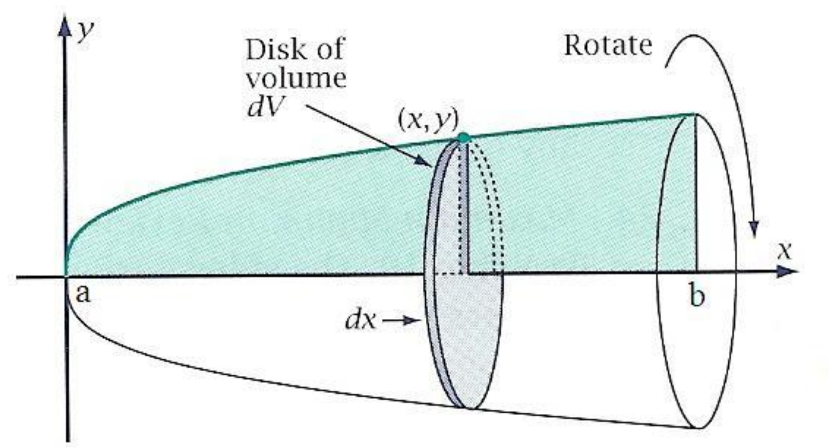
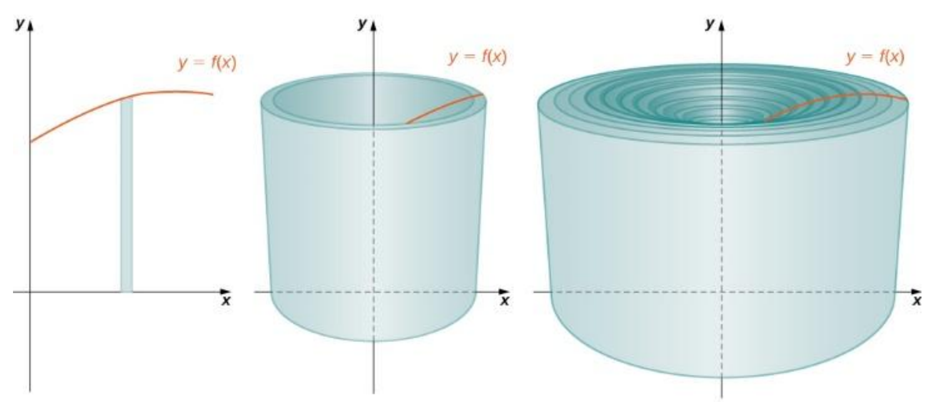
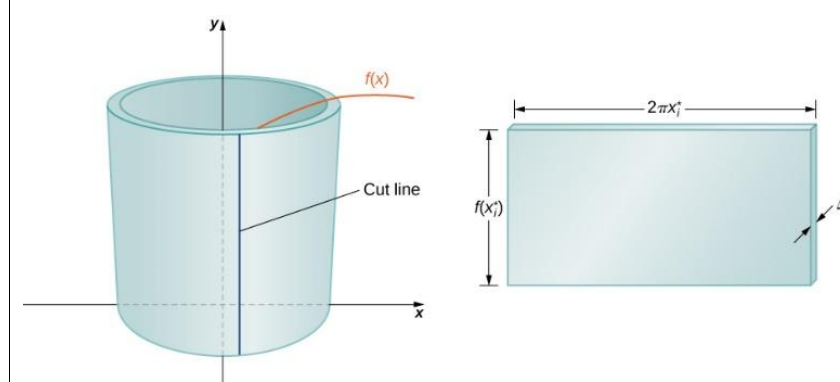
	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Parabola	$y = ax^2 + bx + c$ $y = (x-h)^2 + k$ *Vertical Axis of Symmetry:* $x^2 = 4py$ $(x-h)^2 = 4p(y-k)$ *Vertex:* (h, k) *Focus:* $(h, k+p)$ *Directrix:* $y = k - p$ *Horizontal Axis of Symmetry:* $y^2 = 4px$ $(y-k)^2 = 4p(x-h)$ *Vertex:* (h, k) *Focus:* $(h+p, k)$ *Directrix:* $x = h - p$ *General Form:* $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ *where* $B^2 - 4AC = 0$ *or* $AC = 0$ *If B ≠ 0, then rotate coordinate system:* $\cot 2\theta = \frac{A-C}{B}$ $x = x' \cos\theta - y' \sin\theta$ $y = y' \cos\theta + x' \sin\theta$ *New = (x′, y′), Old = (x, y)* *rotates through angle θ from x-axis*	*Vertical Axis of Symmetry:* $r = \frac{ed}{1 \pm e\cos\theta}$ *Horizontal Axis of Symmetry:* $r = \frac{ed}{1 \pm e\sin\theta}$ *Eccentricity*: $e = 1$ *and* $d = 2p$ 	 ***Interesting Note:*** *The distances from a point on the curve to the focus is the same as to the directrix.*	*Vertical Axis of Symmetry:* $x(t) = 2pt + h$ $y(t) = pt^2 + k$ *(opens upwards)* $y(t) = -pt^2 - k$ *(opens downwards)* $[t_{min}, t_{max}] = [-c, c]$ *Vertex:* (h, k) *Horizontal Axis of Symmetry:* $y(t) = 2pt + k$ $x(t) = pt^2 + h$ *(opens to the right)* $x(t) = -pt^2 - h$ *(opens to the left)* $[t_{min}, t_{max}] = [-c, c]$ *Vertex:* (h, k) *Projectile Motion:* $x(t) = x_0 + v_x t + \left(\frac{1}{2}\right) a_x t^2$ $y(t) = y_0 + v_y t - 16t^2$ *feet* $y(t) = y_0 + v_y t - 4.9t^2$ *meters* $v_x = v\cos\theta$ $v_y = v\sin\theta$ *General Form:* $x = At^2 + Bt + C$ $y = Lt^2 + Mt + N$ *where A and L have the same sign*		$(x\ \ y)\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (D\ \ E)\begin{pmatrix} x \\ y \end{pmatrix} + F = 0$ *where* $\det\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} = 0$
Paraboloid	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \frac{(z-l)^2}{c^2}$					

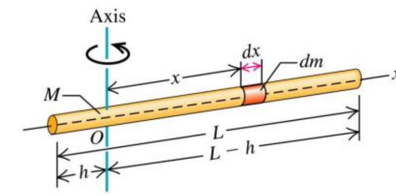
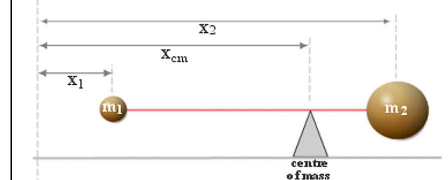
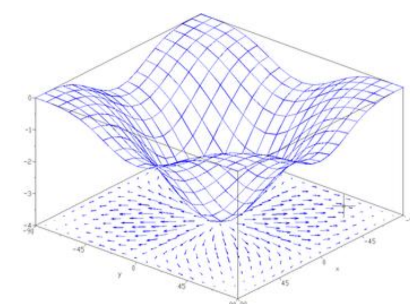
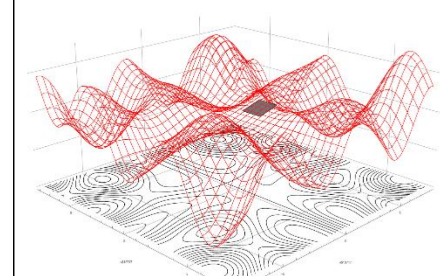
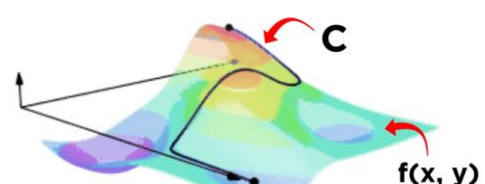
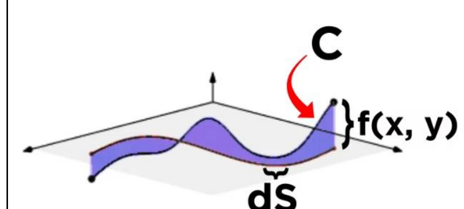
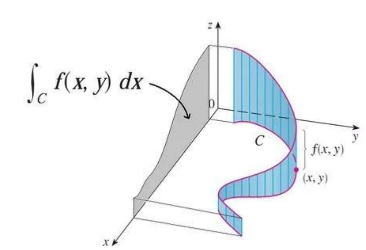
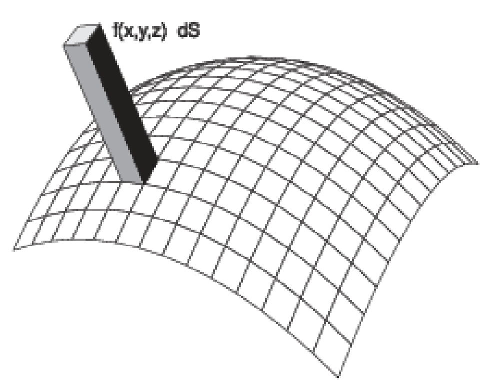
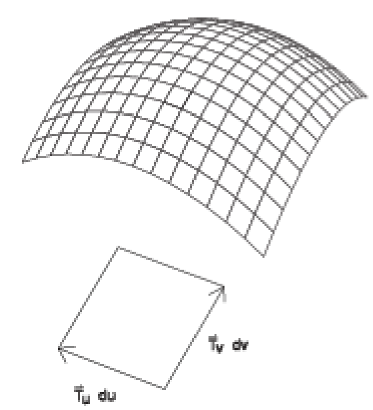
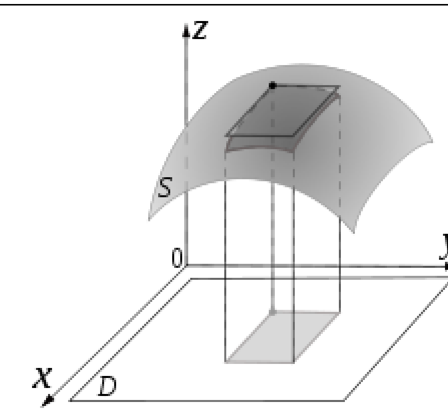
	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Limit	$\lim_{x\to c} f(x) = L$					
1st Derivative	$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ $f'(x) = \frac{dy}{dx} = y' = D_x$	$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$ *Hint: Use Product Rule for* $y = r\sin\theta$ $x = r\cos\theta$		$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, *provided* $\frac{dx}{dt} \neq 0$	$\frac{d}{dt}(\vec{\boldsymbol{r}}) = \vec{\boldsymbol{r}}\,'$ *Unit tangent vector* $\vec{T}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{\lVert\vec{\boldsymbol{r}}'(t)\rVert}$ *where* $\vec{\boldsymbol{r}}'(t) \neq \vec{0}$	$\frac{\partial f}{\partial \boldsymbol{r}} = \frac{\partial f}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial f}{\partial\theta}\mathbf{e}_\theta + \frac{1}{R\sin(\theta)}\frac{\partial f}{\partial\varphi}\mathbf{e}_\varphi$
2nd Derivative	$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y'' = D_{xx}$	$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$		$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}}$	*Unit normal vector* $\vec{N}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{\lVert\vec{\boldsymbol{T}}'(t)\rVert}$ *where* $\vec{\boldsymbol{T}}'(t) \neq \vec{0}$	
Integral	*Fundamental Theorem of Calculus:* $F(x) = \int_a^b f(x)\,dx = F(b) - F(a)$			*Riemann Sum:* $S = \sum_{i-1}^{n} f(y_i)(x_i - x_{i-1})$ *Left Sum:* $S = \left(\frac{1}{n}\right)\left[f(a) + f\left(a+\frac{1}{n}\right) + f\left(a+\frac{2}{n}\right) + \cdots + f\left(b-\frac{1}{n}\right)\right]$ *Middle Sum:* $S = \left(\frac{1}{n}\right)\left[f\left(a+\frac{1}{2n}\right) + f\left(a+\frac{3}{2n}\right) + \cdots + f\left(b-\frac{1}{2n}\right)\right]$ *Right Sum:* $S = \left(\frac{1}{n}\right)\left[f\left(a+\frac{1}{n}\right) + f\left(a+\frac{2}{n}\right) + \cdots + f(b)\right]$	$\int_a^b \vec{\boldsymbol{r}}(t)\,dt = \langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \rangle$	
Double Integral	$\int_a^b \int_{c(y)}^{d(y)} f(x,y)\,dx\,dy$	*Same as rectangular, but* $f(x,y) \longrightarrow f(\rho\cos\phi, \rho\sin\phi)$				

	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Triple Integral	$\int_a^b \int_{c(z)}^{d(z)} \int_{e(y,z)}^{g(y,z)} f(x,y,z)\,dx\,dy\,dz$	*Same as rectangular, but* $f(x,y,z) \longrightarrow f(\rho\cos\phi, \rho\sin\phi, z)$	*Same as rectangular, but* $f(x,y,z) \longrightarrow f(\rho\cos\theta\sin\phi,$ $\rho\sin\theta\sin\phi, \rho\cos\phi)$			
Inverse Functions	*If* $f(x) = y$, *then* $f^{-1}(y) = x$ *Inverse Function Theorem:* $f^{-1}(f'(a)) = \frac{1}{f'(a)}$	*if* $y = \sin\theta$ *then* $\theta = \sin^{-1} y$ *if* $y = \cos\theta$ *then* $\theta = \cos^{-1} y$ *if* $y = \tan\theta$ *then* $\theta = \tan^{-1} y$ *if* $y = \csc\theta$ *then* $\theta = \csc^{-1} y$ *if* $y = \sec\theta$ *then* $\theta = \sec^{-1} y$ *if* $y = \cot\theta$ *then* $\theta = \cot^{-1} y$	*or* $\theta = \arcsin y$ *or* $\theta = \arccos y$ *or* $\theta = \arctan y$ *or* $\theta = \text{arccsc}\, y$ *or* $\theta = \text{arcsec}\, y$ *or* $\theta = \text{arccot}\, y$			
Arc Length	$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$ *Proof:* $\Delta s = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ $ds = \sqrt{dx^2 + dy^2}$ $ds = \sqrt{dx^2 + dy^2\left(\frac{dx^2}{dx^2}\right)}$ $ds = \sqrt{dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2}$ $ds = \sqrt{dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$ $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ $L = \int ds$	*Polar:* $L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$ *Where* $r = f(\theta)$ *Circle:* $L = s = r\theta$ *Proof:* $L = (fraction\ of\ circumference) \cdot$ $\pi \cdot (diameter)$ $L = \left(\frac{\theta}{2\pi}\right)\pi\,(2r) = r\theta$	*C = πd = 2πr* 	*Rectangular 2D:* $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ *Rectangular 3D:* $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$ *Cylindrical:* $L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$ *Spherical:* $L =$ $\int_{t_1}^{t_2} \sqrt{\left(\frac{d\rho}{dt}\right)^2 + \rho^2 \sin^2\varphi\left(\frac{d\theta}{dt}\right)^2 + \rho^2\left(\frac{d\varphi}{dt}\right)^2}$	$L = \int_a^b \lVert\vec{r}'(t)\rVert\,dt$ *s(t) =* $\int_0^t \lVert\vec{r}'(u)\rVert\,du$	
Curvature	$\kappa = \frac{\lvert y''\rvert}{(1 + y'^2)^{3/2}}$	$\kappa(\theta) = \frac{\lvert r^2 + 2r'^2 - rr''\rvert}{(r^2 + r'^2)^{3/2}}$ *for r(θ)*		$\kappa = \frac{\sqrt{(z''y' - y''z')^2 + (x''z' - z''x')^2 + (y''x - x''y')^2}}{(x'^2 + y'^2 + z'^2)^{3/2}}$ *where f(t) = (x(t), y(t), z(t))*	$\kappa = \left\lvert\frac{d\vec{T}}{ds}\right\rvert$ $\kappa = \frac{\lVert\vec{T}'(t)\rVert}{\lVert\vec{r}'(t)\rVert}$ $\kappa = \frac{\lVert\vec{r}'(t) \times \vec{r}''(t)\rVert}{\lVert\vec{r}'(t)\rVert^3}$	*(See Wikipedia: Curvature)*

	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Perimeter	*Square:* $P = 4s$ *Rectangle:* $P = 2l + 2w$ *Triangle:* $P = a + b + c$ *Circle:* $C = \pi d = 2\pi r$ *Ellipse:* $C \approx \pi(a + b)$	*Ellipse:* $C \approx 2\pi\sqrt{\frac{a^2+b^2}{2}}$ $C \approx \pi\left[3(a+b) - \sqrt{(3a+b)(a+3b)}\right]$ $C \approx \pi\,(a+b)\left(1 + \frac{3h}{10 + \sqrt{4-3h}}\right)$	*Ellipse:* $C = 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 sin^2\theta}\, d\theta$ $h = \frac{(a-b)^2}{(a+b)^2}$ & $k^2 = \left(1 - \frac{b^2}{a^2}\right)$			
Area	*Square:* $A = s^2$ *Rectangle:* $A = lw$ *Rhombus:* $A = ½\, ab$ *Parallelogram:* $A = Bh$ *Trapezoid:* $A = \frac{(B_1 + B_2)}{2}\, h$ *Kite:* $A = \frac{d_1 d_2}{2}$ *Triangle:* $A = ½\, Bh$ *Triangle:* $A = ½\, ab \sin(C)$ *Triangle using Heron's Formula:* $A = \sqrt{s(s-a)(s-b)(s-c)}$ $where\ s = \frac{a+b+c}{2}$ *Equilateral Triangle:* $A = ¼\sqrt{3}s^2$ *Frustum:* $A = \frac{1}{3}\left(\frac{B_1+B_2}{2}\right) h$ *Circle:* $A = \pi r^2$ *Circular Sector:* $A = ½\, r^2\theta$ *Ellipse:* $A = \pi ab$	$A = \int_\alpha^\beta \frac{1}{2}[f(\theta)]^2 d\theta$ $where\ \ r = f(\theta)$  *Area of a sector where arc length* $s = r\theta$: $A = \int s\, dr = \int r\theta\, dr = \frac{1}{2}r^2\theta$		$A = \int_\alpha^\beta g(t)\, f'(t)\, dt$ where $f(t) = x$ and $g(t) = y$ *or* *x(t) = f(t) and y(t) = g(t)* *Simplified:* $A = \int_\alpha^\beta y(t)\, \frac{dx(t)}{dt}\, dt$ *Proof:* $\int_a^b f(x)\ dx$ *y = f(x) = g(t)* $dx = \frac{df(t)}{dt} dt = f'(t)\, dt$	$A = \iint_D \left\lvert \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\rvert du\, dv$	
Lateral Surface Area	*Cylinder:* $SA = 2\pi rh$ *Cone:* $SA = \pi rl$ $SA = 2\pi \int_a^b f(x)\, \sqrt{1 + [f'(x)]^2}\, dx$	*For rotation about the x-axis:* $SA = \int 2\pi y\ ds$ *For rotation about the y-axis:* $SA = \int 2\pi x\ ds$ $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$ $r = f(\theta), \quad \alpha \le\ \theta \le \beta$	*Sphere:* $SA = 4\pi r^2$	*For rotation about the x-axis:* $SA = \int 2\pi y\ ds$ *For rotation about the y-axis:* $SA = \int 2\pi x\ ds$ $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$ $if\ x = f(t), y = g(t), \alpha \le\ t \le \beta$		
Total Surface Area	*Cube:* $SA = 6s^2$ *Rectangular Box:* $SA = 2lw + 2wh + 2hl$ *Regular Tetrahedron:* $SA = 2bh$ *Cylinder:* $SA = 2\pi r\,(r + h)$ *Cone:* $SA = \pi r^2 + \pi rl = \pi r\,(r + l)$ *Sphere:* $SA = 4\pi r^2$	*Ellipsoid:* $SA \approx$ $4\pi\left(\frac{a^p b^p + a^p c^p + b^p c^p}{3}\right)^{1/p}$ *Where* $p \approx 1.6075$, $\lvert Relative\ Error\rvert \le 1.061\%$ *(Knud Thomsen's Formula)*	*Ellipsoid:* $S =$	$\int_0^{2\pi}\int_0^{\pi} \text{Sin}[\theta]\ \sqrt{b^2 c^2\, \text{Sin}[\theta]^2\, \text{Cos}[\phi]^2 + a^2 c^2\, \text{Sin}[\theta]^2\, \text{Sin}[\phi]^2 + a^2 b^2\, \text{Cos}[\theta]^2}\ d\theta\, d\phi =$ $2\pi\left(c^2 + \frac{b\, c^2}{\sqrt{a^2 - c^2}}\ \text{EllipticF}[\theta, m] + b\sqrt{a^2 - c^2}\ \text{EllipticE}[\theta, m]\right)$ *where* $m = \frac{a^2\ (b^2 - c^2)}{b^2\ (a^2 - c^2)}$; $\theta = \text{ArcSin}\left[\sqrt{1 - \frac{c^2}{a^2}}\ \right]$; $a \ge b \ge c$		

	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Surface of Revolution	*For revolution about the x-axis:* $A = 2\pi \int_a^b f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ *For revolution about the y-axis:* $A = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$	*For revolution about the x-axis:* $A = 2\pi \int_\alpha^\beta r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ *For revolution about the y-axis:* $A = 2\pi \int_\alpha^\beta r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$	*Sphere:* $S = 4\pi r^2$	*For revolution about the x-axis:* $A_x = 2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ *For revolution about the y-axis:* $A_y = 2\pi \int_a^b x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$		
Volume	*Cube:* $V = s^3$ *Rectangular Prism:* $V = lwh$ *Cylinder:* $V = \pi r^2 h$ *Triangular Prism:* $V = Bh$ *Tetrahedron:* $V = ⅓\, Bh$ *Pyramid:* $V = ⅓\, Bh = ⅓\, lwh$ *Cone:* $V = ⅓\, Bh = ⅓\, \pi r^2 h$ *Sphere:* $V = \frac{4}{3}\pi r^3$ *Ellipsoid:* $V = \frac{4}{3}\pi abc$ $\int\int\int f(x,y,z)\, dx\, dy\, dz$	$\int\int\int f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta$	$\int\int\int f\begin{pmatrix}\rho\sin\varphi\cos\theta, \\ \rho\sin\varphi\sin\theta, \\ \rho\cos\varphi\end{pmatrix}$ … $\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$			*Ellipsoid:* $V = \frac{4}{3}\pi\sqrt{det(A^{-1})}$

	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Volume of Revolution	**Disk Method** $V = \int_a^b (area\ of\ circle)\ d(thickness)$ *Rotation about the x-axis:* $V = \int_a^b \pi\,[f(x)]^2 dx$ *Rotation about the y-axis:* $V = \int_c^d \pi x^2 dy$  					
	Washer Method *Rotation about the x-axis:* $V = \int_a^b \pi\,\{[f(x)]^2 - [g(x)]^2\}\,dx$ $V = V_{Outer\ Disk} - V_{Inner\ Disk}$					
	Shell Method $V = \int_a^b (circumference)\ (hight)\ dx$ *Rotation about the y-axis:* $V = \int_a^b 2\pi x\ f(x)\ dx$ *Rotation about the x-axis:* $V = \int_c^d 2\pi y\ g(y)\ dy$  (a) (b) (c)			 (a) (b)		

	Rectangular	Polar/Cylindrical	Spherical	Parametric	Vector	Matrix
Moment of Inertia	$I = \sum_{i=1}^{N} m_i\, r_i^2 = \int_0^a m\, r^2\, dr$	$J = \int_A r^2\, dA = \int_A (x^2 + y^2)\, dA$ $J = I_x + I_y$			$I = \iiint_V \rho(\boldsymbol{r})\; d(\boldsymbol{r})^2\; dV(\boldsymbol{r})$	*(see Wikipedia: Moment of Inertia)*
Center of Mass	$\boldsymbol{R} = \frac{1}{M}\sum_{i=1}^{N} m_i\, r_i$ *where* $M = \sum_{i=1}^{N} m_i$ *1D for Discrete:* $x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$	*2D for Discrete:* $M_y = \sum_{i=1}^{N} m_i\, x_i$ $M_x = \sum_{i=1}^{N} m_i\, y_i$ $\bar{x} = \frac{M_y}{M},\; \bar{y} = \frac{M_x}{M}$	*3D for Discrete:* $x_{cm} = \bar{x} = \frac{1}{M}\sum_{i=1}^{N} m_i\, x_i$ $y_{cm} = \bar{y} = \frac{1}{M}\sum_{i=1}^{N} m_i\, y_i$ $z_{cm} = \bar{z} = \frac{1}{M}\sum_{i=1}^{N} m_i\, z_i$	*3D for Continuous:* $\bar{x} = \frac{1}{M}\int_0^M x\, dm$ $\bar{y} = \frac{1}{M}\int_0^M y\, dm$ $\bar{z} = \frac{1}{M}\int_0^M z\, dm$ *where* $M = \int_0^M dm$ *and* $dm = \rho\, dz\, dy\, dx$	$\boldsymbol{R} = \frac{1}{M}\int \boldsymbol{r}\, dm$ $\boldsymbol{R} = \frac{1}{M}\iiint_V \rho(\boldsymbol{r})\, \boldsymbol{r}\, dV$ *Where* $\boldsymbol{r}$ *is distance from the axis of rotation, not origin.*	
Gradient	$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$	$\nabla f(\rho, \phi, z) =$ $\frac{\partial f}{\partial \rho}\boldsymbol{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi + \frac{\partial f}{\partial z}\boldsymbol{e}_z$	$\nabla f(r, \theta, \phi) =$ $\frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta + \frac{\mathbf{1}}{r\,\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi$		$(\nabla f(x)) \bullet \boldsymbol{v} = D_{\boldsymbol{v}} f(x)$ $\nabla f = \frac{\partial f_i}{\partial x_j}\boldsymbol{e}_i\boldsymbol{e}_j$ *where* $f = (f_1, f_2, f_3)$	
Line Integral (Contour Integral if Complex)	$\int_C f(x, y)\, ds =$ $\int_a^b f(x, y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$				$\int_C f\, ds = \int_a^b f(\boldsymbol{r}(t))\; \lvert\boldsymbol{r}'(t)\rvert\, dt$ $\int_C \boldsymbol{F}(r) \bullet dr = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \bullet \boldsymbol{r}'(t)dt = \boldsymbol{F}(b) - \boldsymbol{F}(a)$ *Fundamental Theorem for Line Integrals*	
Surface Integral	$\int_S f\, dS =$ $\iint_T f(\boldsymbol{x}(s,t))\left\lvert\frac{\partial \boldsymbol{x}}{\partial s} \times \frac{\partial \boldsymbol{x}}{\partial t}\right\rvert ds\, dt$ *Where* $\boldsymbol{x}(s,t) =$ $(x(s,t), y(s,t), z(s,t))$ *and* $\left(\frac{\partial \boldsymbol{x}}{\partial s} \times \frac{\partial \boldsymbol{x}}{\partial t}\right) =$ $\left(\frac{\partial(y,z)}{\partial(s,t)}, \frac{\partial(z,x)}{\partial(s,t)}, \frac{\partial(x,y)}{\partial(s,t)}\right)$				$\int_S \boldsymbol{v} \bullet d\boldsymbol{S} =$ $\int_S (\boldsymbol{v} \bullet \boldsymbol{n})\, d\boldsymbol{S} =$ $\iint_T \boldsymbol{v}(\boldsymbol{x}(s,t))$ $\bullet \left(\frac{\partial \boldsymbol{x}}{\partial s} \times \frac{\partial \boldsymbol{x}}{\partial t}\right) ds\, dt$