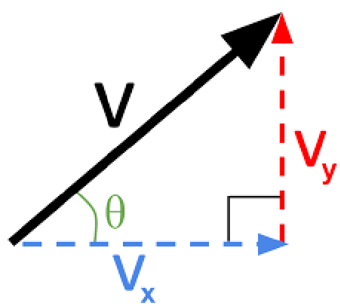
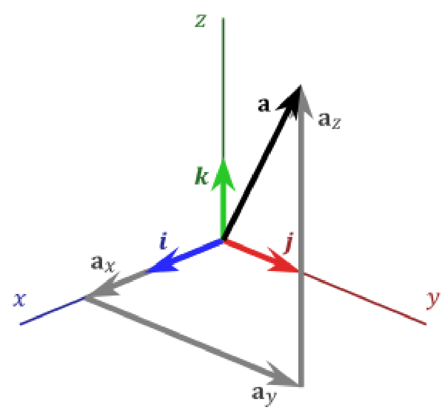
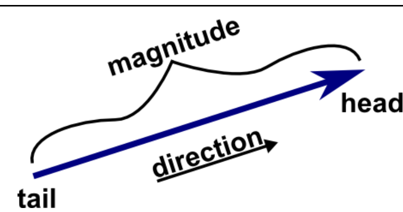
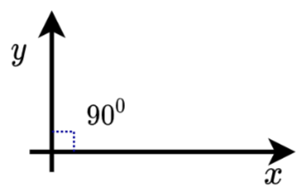
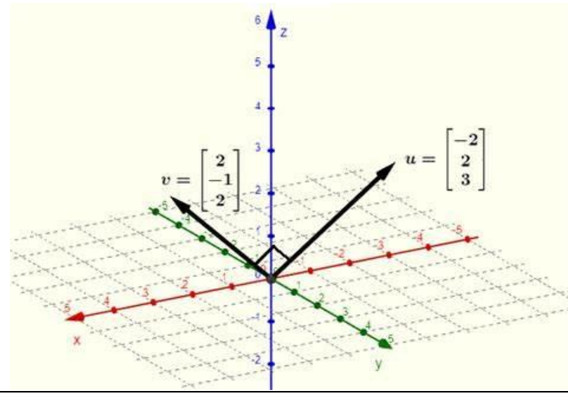
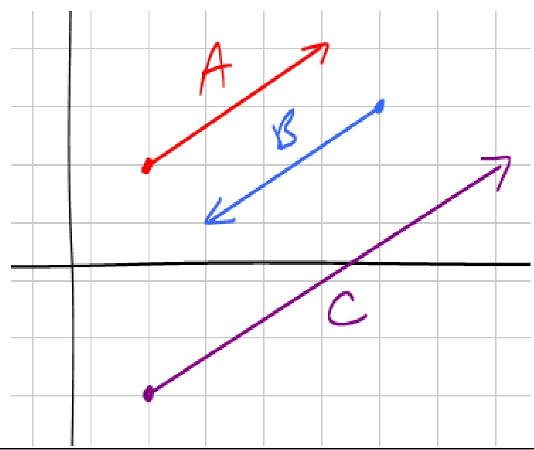
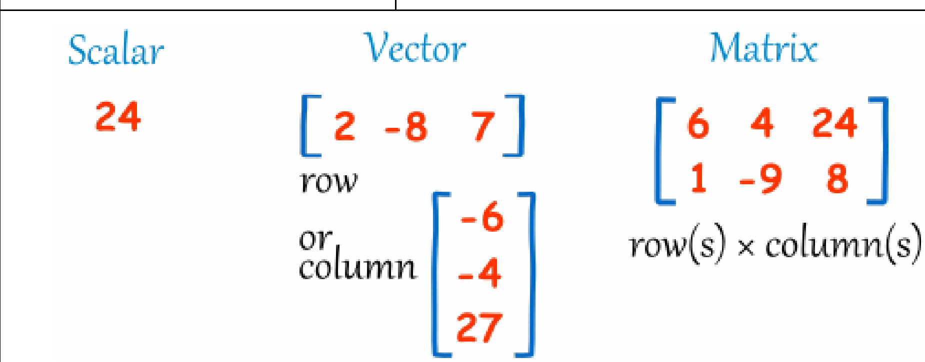
# Harold's Vectors Cheat Sheet

5 December 2022

## Definitions

| Term | Formula | Example |
|---|---|---|
| **Vector Notation** | $\mathbf{A}, \boldsymbol{a}$ | Bold letter |
| | $\vec{a}, \vec{\boldsymbol{a}}$ | Arrow on top |
| **Component Notation** | $\boldsymbol{a} = \vec{a} = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}$ | $\boldsymbol{a} = 3\hat{\imath} + 4\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}$ |
| | $\boldsymbol{a} = \langle a_x, a_y, a_z \rangle$ | $\boldsymbol{a} = \langle 3, 4, 5 \rangle$ |
| | $\boldsymbol{a} = r\angle\theta$ | $\boldsymbol{a} = 5 \angle 53.13°$ (2D) |
| | <br>**2D** | <br>**3D** |
| **Vectors Used in Examples** | $\boldsymbol{a} = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}$<br>$\boldsymbol{b} = b_x\hat{\boldsymbol{\imath}} + b_y\hat{\boldsymbol{\jmath}} + b_z\hat{\boldsymbol{k}}$<br>**2D**: set $a_z = b_z = 0$ | $\boldsymbol{a} = 3\hat{\imath} + 4\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}$<br>$\boldsymbol{b} = 6\hat{\imath} - 7\hat{\boldsymbol{\jmath}} - 8\hat{\boldsymbol{k}}$ |
| **Magnitude** | $\lVert\boldsymbol{a}\rVert = \sqrt{a_x^2 + a_y^2 + a_z^2}$ | $\lVert\boldsymbol{a}\rVert = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$ |
| | Can also use $\lvert\boldsymbol{a}\rvert$.<br>Length of vector, but with no direction (scalar).<br>Similar to a hypotenuse.<br>Think multi-dimensional Pythagorean Theorem. |  |
| **Direction** | Divided into dimensional components. | A scalar with a direction is a vector.<br>Example: speed vs. velocity |
| | $\tan\theta = \frac{a_y}{a_x}$ | $\theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\left(\frac{4}{3}\right) \cong 53.13°$ |
| **Unit Vector** (Basis Vector) | $\hat{\boldsymbol{\imath}} = x - axis = \langle 1, 0, 0 \rangle$<br>$\hat{\boldsymbol{\jmath}} = y - axis = \langle 0, 1, 0 \rangle$<br>$\hat{\boldsymbol{k}} = z - axis = \langle 0, 0, 1 \rangle$ | Circumflex or "hat" on top.<br>Indicates direction only.<br>Always has a magnitude of one (1 or unit). |
| | $\vec{\boldsymbol{u}} = \frac{\boldsymbol{a}}{\lVert\boldsymbol{a}\rVert}$ | $\vec{\boldsymbol{u}} = \frac{a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$ |

<table>
<tr><td>Scalar</td><td>k, m</td><td>A number with no direction or units.</td></tr>
<tr><td rowspan="2">Orthogonal</td><td>A change in one dimension does not change in any of the values in the other dimensions.</td><td>2D: right angle<br>Rectangular Coordinates: The x-axis, y-axis, and z-axis are orthogonal to each other.<br>Polar Coordinates: The angle is orthogonal to the line segment length</td></tr>
<tr><td>if $\boldsymbol{a} \bullet \boldsymbol{b} = 0$</td><td>Two vectors are orthogonal if their dot product is zero.</td></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td>Parallel</td><td>If $\boldsymbol{a} = k\boldsymbol{b}$</td><td>Two vectors are parallel if they have the same direction.</td></tr>
<tr><td></td><td>Collinear of in opposite directions</td><td></td></tr>
<tr><td>Vector vs. Matrix</td><td>vector = $1 \ x \ n$ or $n \ x \ 1$ matrix</td><td>A matrix with only one (1) row or column.</td></tr>
<tr><td></td><td colspan="2"></td></tr>
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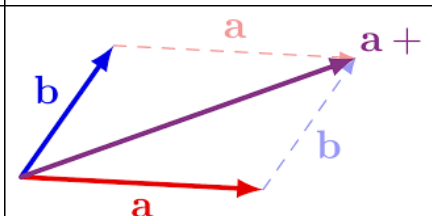
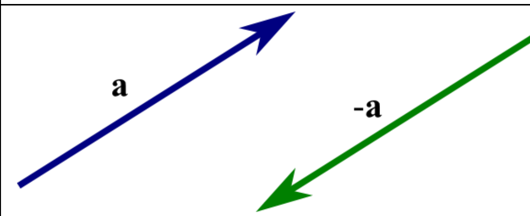
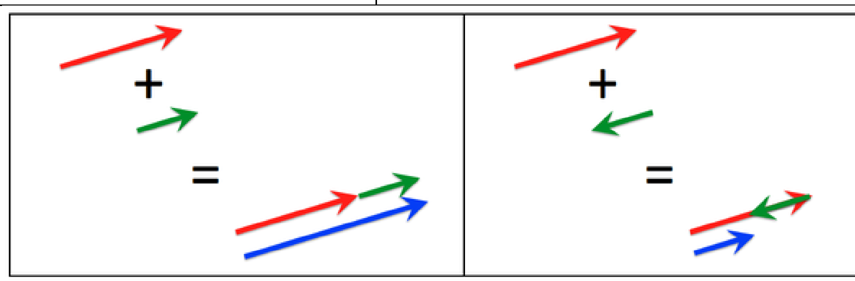
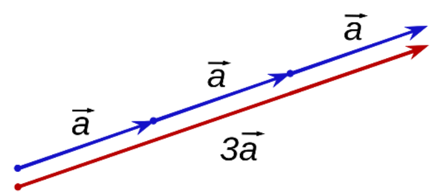
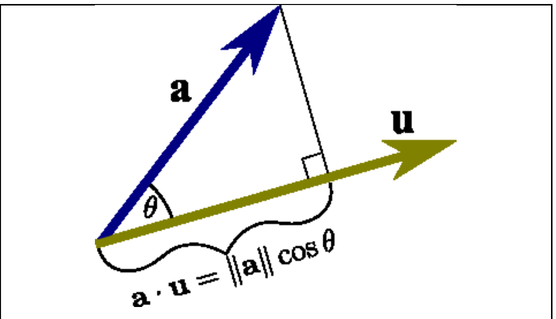
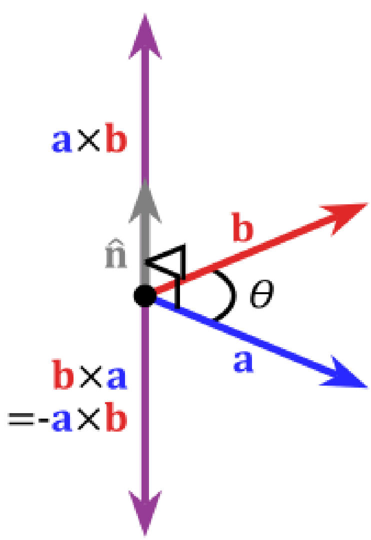
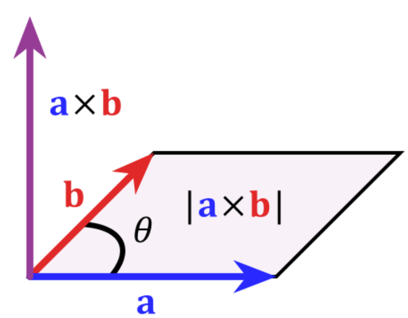
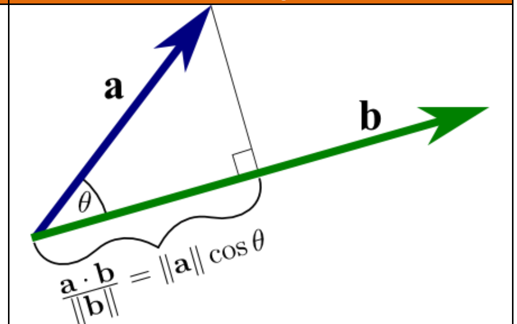
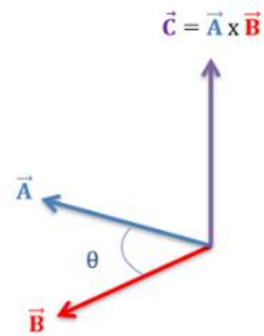
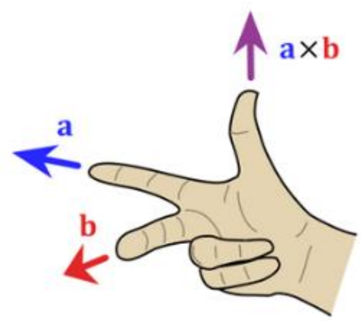
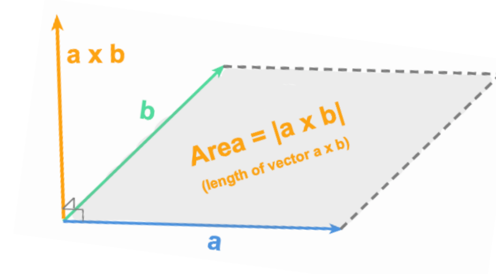
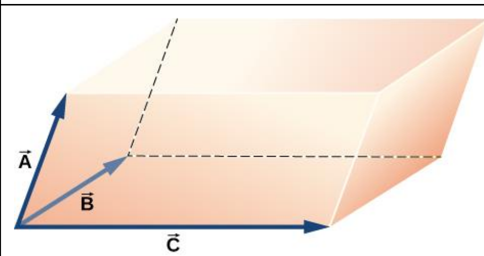
## Vector Operations

| Operation | Formula | Example |
|---|---|---|
| **Addition** | $\boldsymbol{a}+\boldsymbol{b}=(a_x+b_x)\hat{\boldsymbol{\imath}}+(a_y+b_y)\hat{\boldsymbol{\jmath}}+(a_z+b_z)\hat{\boldsymbol{k}}$ | $\boldsymbol{a}+\boldsymbol{b}=(3+6)\hat{\boldsymbol{\imath}}+(4-7)\hat{\boldsymbol{\jmath}}+(5-8)\hat{\boldsymbol{k}}=9\hat{\boldsymbol{\imath}}-3\hat{\boldsymbol{\jmath}}-3\hat{\boldsymbol{k}}$ |
| | $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$ | Commutative |
| | $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$ | Associative |
| | $(k+m)\boldsymbol{a}=k\boldsymbol{a}+m\boldsymbol{a}$ | Distributive |
| | $k(\boldsymbol{a}+\boldsymbol{b})=k\boldsymbol{a}+k\boldsymbol{b}$ | |
| |   |   |
| **Subtraction** | $\boldsymbol{a}-\boldsymbol{b}=\boldsymbol{a}+(-\boldsymbol{b})$ | Change the direction of $\vec{b}$ then add. |
| |   | |
| **Scalar Multiplication** | $k\boldsymbol{a}=ka_x\hat{\boldsymbol{\imath}}+ka_y\hat{\boldsymbol{\jmath}}+ka_z\hat{\boldsymbol{k}}$ | $3\cdot\boldsymbol{a}=3\cdot3\hat{\imath}+3\cdot4\hat{\jmath}+3\cdot5\hat{\boldsymbol{k}}=9\hat{\imath}+12\hat{\jmath}+15\hat{\boldsymbol{k}}$ |
| | Changes the magnitude only. | |
| | $k(\boldsymbol{a}+\boldsymbol{b})=k\boldsymbol{a}+k\boldsymbol{b}$ |   |
| **Dot Product** (Scalar Product) | $\boldsymbol{a}\bullet\boldsymbol{b}=a_xb_x+a_yb_y+a_zb_z$ | $\boldsymbol{a}\bullet\boldsymbol{b}=(3)(6)+(4)(-7)+(5)(-8)=-50$ |
| | $\boldsymbol{a}\bullet\boldsymbol{b}=\lVert\boldsymbol{a}\rVert\,\lVert\boldsymbol{b}\rVert\cos\theta$ | $\cos\theta=\dfrac{\boldsymbol{a}\bullet\boldsymbol{b}}{\lVert\boldsymbol{a}\rVert\,\lVert\boldsymbol{b}\rVert}$ |
| | $\boldsymbol{a}\bullet\boldsymbol{b}=\boldsymbol{b}\bullet\boldsymbol{a}$ | Commutative |
| | $\boldsymbol{a}\bullet(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\bullet\boldsymbol{b}+\boldsymbol{a}\bullet\boldsymbol{c}$ | Distributive |
| | $k(\boldsymbol{a}\bullet\boldsymbol{b})=k\boldsymbol{a}\bullet\boldsymbol{b}=\boldsymbol{a}\bullet k\boldsymbol{b}$ | Scalar Multiplication |
| | $0\bullet\boldsymbol{u}=0$ | Zero Vector Dot Product |
| | $\boldsymbol{u}\bullet\boldsymbol{u}=\lVert\boldsymbol{u}\rVert^2$ | Dot Product and Vector Magnitude Relationship |
| | Is always a scalar. | |

| | | |
|---|---|---|
| | |  |
| **Cross Product** (Vector Product) | $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \boldsymbol{k}$ <br> $= (a_y b_z - b_y a_z)\, \boldsymbol{i} - (a_x b_z - b_x a_z)\, \boldsymbol{j} + (a_x b_y - b_x a_y)\, \boldsymbol{k}$ | |
| | $\lVert \boldsymbol{a} \times \boldsymbol{b} \rVert = \lVert \boldsymbol{a} \rVert \, \lVert \boldsymbol{b} \rVert \sin\theta$ | $\sin\theta = \dfrac{\lVert \boldsymbol{a} \times \boldsymbol{b} \rVert}{\lVert \boldsymbol{a} \rVert \, \lVert \boldsymbol{b} \rVert}$ |
| | $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$ | Anti-Commutative |
| | $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} \neq \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ <br> $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \bullet \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \bullet \boldsymbol{b})\, \boldsymbol{c}$ | Not Commutative |
| | $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ <br> $\boldsymbol{a} \bullet (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \bullet \boldsymbol{c}$ | Not Associative |
| | $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$ <br> $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}$ | Distributive |
| | $k(\boldsymbol{a} \times \boldsymbol{b}) = k\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times k\boldsymbol{b}$ <br> $(k\boldsymbol{a}) \times \boldsymbol{b} = k(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \times (k\boldsymbol{b})$ | Scalar Multiplication |
| | Is always a vector orthogonal to the other two vectors. | |
| |  |  |
| **Scalar Triple Product** | $(\boldsymbol{a} \times \boldsymbol{b}) \bullet \boldsymbol{c} = \boldsymbol{a} \bullet (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \bullet (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \bullet (\boldsymbol{a} \times \boldsymbol{b}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$ | |
| **Vector Triple Product** | $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \bullet \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \bullet \boldsymbol{b})\, \boldsymbol{c}$ | |

## Vector Applications

| Application | Formula | Example |
|---|---|---|
| **Projection** | $proj_{\boldsymbol{a}}\boldsymbol{b} = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{a}}\right)\boldsymbol{a}$ |   |
| **Right Hand Rule** | The cross product produces a vector orthogonal to the other two vectors. | Use the right hand rule to determine direction of the cross product vector. |
| |     | |
| **Area** (Parallelagram) | $A = \lVert\boldsymbol{a}\times\boldsymbol{b}\rVert$ |   |
| **Volume** (Parallelepiped) | $V = \lVert(\boldsymbol{a}\times\boldsymbol{b})\bullet\boldsymbol{c}\rVert$ |   |
| **Torque** | $\boldsymbol{\tau} = \boldsymbol{r}\times\boldsymbol{F}$ | $\lVert\boldsymbol{\tau}\rVert = r\,F\sin\theta$ |
| **Coplanar** | Three vectors are coplanar if $(\boldsymbol{a}\times\boldsymbol{b})\bullet\boldsymbol{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = 0$ | All three vectors are in the same plane. |