Harold's Simplex Tableau Cheat Sheet
(Linear Optimization)

28 April 2022

<table>
<tr><th colspan="2">How to Optimize using the Simplex Method</th></tr>
<tr><td>Steps</td><td>
1. Read the word problem at least 4 times
2. Assign non-basic variables (x_1, x_2, …)
3. List optimization function, z = _______, that will be maximized
4. List inequalities (constraints)
5. Add basic variables, also called slack variables, (s_1, s_2, …), to turn inequalities into equations
    a. ≤ means s_n is positive (default)
    b. ≥ means s_n is negative (seldom used)
    c. Column has all zeros (0) except for one (1) for the slack variable
6. Organize the equation and inequalities into a matrix, with variables for the columns
7. Construct a simplex tableau corresponding to the system
    a. Rows 1-n are the inequalities
    b. Last row (indicator row) is the z equation solved to equal zero (0)
        i. Example: if $z = 5x_1 + 7x_2$, then $-5x_1 -7x_2 + z = 0$, or -5 -7 1 | 0
8. If the indicator row coefficients are all positive, then the problem is solved, otherwise …
9. Find pivot
    a. Pivot Column is the most <u>negative</u> value in indicator row on bottom
    b. Pivot Row is the smallest <u>positive</u> ratio of pivot column coefficient to b value on far right
10. Pivot (perform matrix row operations) to create a new simplex tableau
    a. Example: $R_1 = R_1 – 2R_2$
    b. All values in column should be turned into zeros (0) except the pivot element (like the Identity matrix)
    c. Pivot element should be turned into one (1) using division <u>afterwards</u> to avoid working with fractions
    d. Column b should always be positive when maximizing
11. Repeat steps 8 - 10 until no more negatives in the indicator row on bottom
12. Maximum objective function value is in the simplex tableau's bottom right corner
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<tr><td>Example</td><td>
Objective Function:
    $z = x_1 + 2x_2 – x_3$
Subject To:
    $2x_1 + x_2 + x_3 \leq 14$
    $4x_1 + 2x_2 + 3x_3 \leq 28$
    $2x_1 + 5x_2 + 5x_3 \leq 30$
    $x_1 \geq 0; x_2 \geq 0; x_3 \geq 0$
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<td>Simplex Tableau</td>
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Adding slack variables gives:
$2x_1 + x_2 + x_3 + s_1 = 14$
$4x_1 + 2x_2 + 3x_3 + s_2 = 28$
$2x_1 + 5x_2 + 5x_3 + s_3 = 30$
where all variables $x_n \geq 0$ (e.g., not negative)

Simplex Tableau before Pivoting:

$$\begin{matrix} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & | & b \\ R_1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & | & 14 \\ R_2 & 4 & 2 & 3 & 0 & 1 & 0 & 0 & | & 28 \\ R_3 & 2 & \enclose{circle}{5} & 5 & 0 & 0 & 1 & 0 & | & 30 \\ \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} \\ R_4 & -1 & -2 & 1 & 0 & 0 & 0 & 1 & | & 0 \end{matrix}$$

Pivot Determination:
The -2 is the most negative on the bottom row, so pivot column is 2.
Ratios are row 1: 14/1 = 14, row 2: 28/2 = 14, row 3: 30/5 = 6.
The smallest positive ratio is 6.
So, the pivot is at column 2, row 3 = ⑤.
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<td>After Pivot #1</td>
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Row Operations:
Pivot element is Col 2, Row 3.
$R_1 = 5 R_1 – R_3$
$R_2 = 5 R_2 – 2 R_3$
$R_4 = 5 R_4 + 2 R_3$
$R_3 = (1/5) R_3$

Simplex Tableau after Pivot #1:

$$\begin{matrix} & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & | & b \\ R_1 & \enclose{circle}{8} & 0 & 0 & 5 & 0 & -1 & 0 & | & 40 \\ R_2 & 16 & 0 & 5 & 0 & 5 & -2 & 0 & | & 80 \\ R_3 & \frac{2}{5} & 1 & 1 & 0 & 0 & \frac{1}{5} & 0 & | & 6 \\ \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} \\ R_4 & -1 & 0 & 15 & 0 & 0 & 2 & 5 & | & 60 \end{matrix}$$

Pivot Determination:
The -1 is the most negative on the bottom row, so pivot column is 1.
Ratios are row 1: 40/8 = 5, row 2: 80/16 = 5, row 3: 6/(2/5) = 15.
The smallest positive ratio is 5.
So, the pivot is at column 1, row 1 = ⑧. Row 2 also works.
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<td>After Pivot #2</td>
<td colspan="2">Next Pivot element is Col 1, Row 2.
$R_1 = 2\ R_1 - R_2$
$R_3 = 16\ R_3 - (2/5)\ R_2$
$R_4 = 16\ R_4 + R_2$
$R_2 = (1/16)\ R_2$

Final Tableau after Pivot #2:

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ \overline{} \\ R_4 \end{array}\left[\begin{array}{ccccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & | & b \\ 1 & 0 & 0 & \frac{5}{8} & -5 & 0 & -\frac{1}{8} & | & 5 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & | & 0 \\ 0 & 1 & 1 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & | & 4 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 3 & \frac{1}{8} & 0 & \frac{3}{8} & 0 & | & 13 \end{array}\right]$$

Note: All indicators in bottom row are now zero or larger.
13 is not an indicator. It is the maximum solution.</td>
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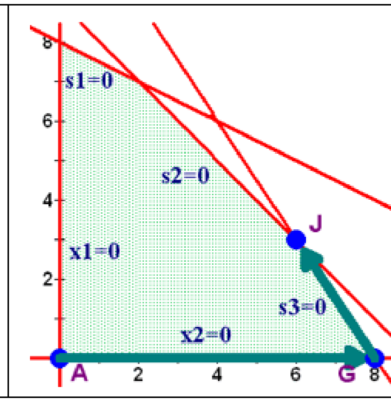
<td>Basic Feasible Solution</td>
<td>$x_1 = 5$ Choose 5 x_1s
$x_2 = 4$ Choose 4 x_2s
$x_3 = 0$ Choose no x_3s
$s_1 = 0$
$s_2 = 0$
$s_3 = 0$
$z = 13$ Objective function value of 13.

Since all slack variables $s_n \geq 0$, this solution is optimal.</td>
<td>
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Sources: https://math.uww.edu/~mcfarlat/s-prob.htm
See also: http://simplex.tode.cz/en/#steps

