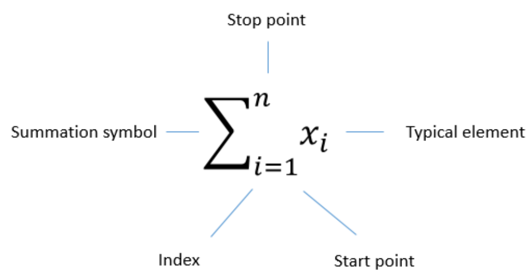
Harold's Series Cheat Sheet

16 March 2024

Sigma Notation



$$\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4$$

Sequence	$\lim_{n\to\infty} a_n = L$	$a_n, a_{n+1}, a_{n+2}, \dots$
Series	$\sum_{n=1}^{\infty} a_n = \mathbf{S}$	$a_n + a_{n+1} + a_{n+2} + \cdots$

Recursive and Explicit

Operation	Arithmetic Series	Geometric Series
Summation Notation	$S_n = \sum_{k=1}^{n} a_k$	$S_n = \sum_{k=0}^{n-1} a_0 r^k = \sum_{k=1}^{n} a_0 r^{k-1}$
Summation Expanded	$S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$	$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}$
Sum of n Terms (Finite Series)	$S_n = \frac{n}{2}(a_1 + a_n)$ $S_n = \frac{n}{2}[2a_1 + (n-1)d]$	$S_n = a_1 \frac{(1 - r^n)}{1 - r}$ $S_n = \frac{a_1 - a_n r}{1 - r}$
Sum of ∞ Terms (Infinite Series)	$S_\infty \to \infty$	$S_\infty = \frac{a_1}{1 - r}\ if\ \lvert r \rvert < 1$
Recursive nth Term	$a_n = a_{n-1} + d$	$a_n = a_{n-1} r$
Explicit nth Term	$a_n = a_1 + (n-1)d$	$a_n = a_1 r^{n-1}$

Summation Formulas

Type	Summation Formulas
Constant Multiple Rule	$\sum_{i=1}^{n} c \cdot a_i = c\sum_{i=1}^{n} a_i$
Sum Rule	$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
Change of Bounds	$\sum_{i=m}^{n} a_i = \sum_{i=p}^{p+n-m} a_{i+m-p}$
Sum of Powers (Arithmetic Series)	$\sum_{i=1}^{n} c = cn$ $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$ $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2 = \frac{n^2(n+1)^2}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$ $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$ $\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12} = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}$ $\sum_{i=1}^{n} i^6 = \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}$ $\sum_{i=1}^{n} i^7 = \frac{n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)}{24}$ $S_k(n) = \sum_{i=1}^{n} i^k = \frac{(n+1)^{k+1}}{k+1} - \frac{1}{k+1}\sum_{r=0}^{k-1}\binom{k+1}{r} S_r(n)$

Interesting Summation Formulas	$\sum_{i=1}^{n} i(i+1) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \frac{n(n+1)(n+2)}{3}$ $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ $\sum_{i=1}^{n} 2i - 1 = n^2$ $\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ $\sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

Binomial Theorem

Binomial Series		Expanded	
Pascal's Triangle	1 1 1 1 2 1 1 3 3 1 1 4 6 4 1 1 5 10 10 5 1	${}_0C_0$ ${}_1C_0$ ${}_1C_1$ ${}_2C_0$ ${}_2C_1$ ${}_2C_2$ ${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$ ${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$ ${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$	$\binom{0}{0}$ $\binom{1}{0}$ $\binom{1}{1}$ $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$ $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$
Example	$(a \pm b)^0 =$ $(a \pm b)^1 =$ $(a \pm b)^2 =$ $(a \pm b)^3 =$ $(a \pm b)^4 =$ $(a \pm b)^5 =$	1 $a \pm b$ $a^2 \pm 2ab + b^2$ $a^3 \pm 3a^2b + 3ab^2 \pm b^3$ $a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$ $a^5 \pm 5a^4b + 10a^3b^2 \pm 10a^2b^3 + 5ab^4 \pm b^5$	
Binomial Theorem	$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k =$ $\binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \boldsymbol{\binom{n}{r} a^{n-r} b^r} + \cdots + \binom{n}{n-1} ab^{n-1} + \binom{n}{n} b^n$		
$(1+x)^r = \sum_{n=0}^{+\infty} \binom{r}{n} x^n$		$(1+x)^r = 1 + \sum_{n=1}^{+\infty} \frac{r(r-1)(r-2)\ldots(r-n+1)}{n!} x^n$ $= 1 + rx + \frac{r(r-1)}{2!} x^2 + \frac{r(r-1)(r-2)}{3!} x^3 + \cdots$	

Factorials and Constants

Operation	Formula
Factorial	$n! = n \bullet (n-1) \bullet (n-2) \bullet \ldots \bullet 3 \bullet 2 \bullet 1$
Double Factorial	$n!! = n \bullet (n-2) \bullet (n-4) \bullet \ldots \bullet 4 \bullet 2$ (Even n) $n!! = n \bullet (n-2) \bullet (n-4) \bullet \ldots \bullet 3 \bullet 1$ (Odd n)
Gamma Function (Continuous Factorial)	$\Gamma(n+1) = n\,\Gamma(n) = n!$ $\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt$
Combination	${}_nC_r = \frac{n!}{r!\,(n-r)!}$ $= \binom{n}{r} = \prod_{k=1}^{r} \frac{n-k+1}{k} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$ *Converges for* $\lvert x \rvert < 1$ *and all complex r , r ≠ 0, where*
Permutation	${}_nP_r = \frac{n!}{(n-r)!}$
Fibonacci Sequence	Recursive: $F_0 = 0$ $F_1 = 1$ $\mathbf{F_n = F_{n-1} + F_{n-2}}$ Explicit: $\mathbf{F}_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right), \quad n \in \mathbb{N}$ **F** = {0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, …}
Golden Ratio	$\varphi = \frac{1+\sqrt{5}}{2} \cong \frac{\mathbf{F}_n}{\mathbf{F}_{n-1}}$ $\mathbf{F}_n \cong \frac{\varphi^n + (1-\varphi)^n}{\sqrt{5}}$ $\varphi \cong$ 1.6180 33988 74989 48482 04586 83436 56381 17720 30917 98057 …
Euler's Identity	$e^{i\pi} + 1 = 0$ *Since* $x = \pi$ *in* $e^{ix} = cos(x) + i \cdot sin(x)$
Euler's Number	$e \cong$ 2.71828 18284 59045 23536 02874 71352 66249 77572 47093 69995 ….
Imaginary Unit	$i = \sqrt{-1} = 0 + 1i$
Archimedes' Constant (pi)	π ≅ 3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 … $\pi \cong \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{52{,}163}{16{,}604}, \frac{103{,}993}{33{,}102}, \frac{104{,}348}{33{,}215}, \frac{245{,}850{,}922}{78{,}256{,}779}$

