Harold's Proofs Cheat Sheet

22 October 2022

Definitions

Term	Example	Definition
Proof	*The sum of two even integers is always even. …*	Exhaustive deductive reasoning which establishes logical certainty for *all* cases.
Theorem	$a^2 + b^2 = c^2$ *for a right-angled triangle*	A non-self-evident statement that has been proven to be true, either on the basis of generally accepted statements such as axioms or on the basis of previously established statements such as other theorems.
Axioms	*"Nothing can both be and not be at the same time and in the same respect"*	An axiom, postulate or assumption is a statement that is taken to be true, to serve as a premise or starting point for further reasoning and arguments.
Conjecture	*When n is a prime number; n + 2 is always prime.*	A conclusion or a proposition which is suspected to be true due to preliminary supporting evidence, but for which no proof or disproof has yet been found.
Hypothesis	*Drinking sugary drinks daily leads to being overweight*	A proposed explanation for a phenomenon. Antecedent. P is the assumption in a (possibly counterfactual) "What If?" question.
Proposition	*P* = "the sky is Purple."	A statement that is either true or false
Statement	**Let** <insert hypothesis>. **If** <insert hypothesis>, **then** <insert conclusion>.	
Antecedent	$P \to Q$, *P is the antecedent*	Assumptions or premises of a conditional statement.
Consequent	$P \to Q$, *Q is the consequent*	Conclusions of a conditional statement.
Free Variable	$y \in \{x \mid x2 < 9\}$, *y is a free variable*	• Letters that stand for objects that the statement says something about. • They stand for some particular but unspecified elements of the universe of discourse. • x is free to stand for anything.
Bound (Dummy) Variable	$y \in \{x \mid x2 < 9\}$, *x is a bound variable*	• Letters that are used as a convenience to help express an idea and should not be thought of as standing for any particular object. • A bound variable can always be replaced by a new variable without changing the meaning of the statement.
Instance	$x = 4$	• An assignment of particular values to free variables.
Tautologies	$P \lor \neg P$	Formulas that are always true.
Contradictions	$P \land \neg P$	Formulas that are always false.

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Proof Methods

<table>
<tr><th>Method</th><th>Definition</th></tr>
<tr><td>Direct</td><td>The conclusion is established by logically combining the axioms, definitions, and earlier theorems.
When given $P \rightarrow Q$, assume P is true, then prove Q.</td></tr>
<tr><td>Contrapositive</td><td>Infers the statement $P \rightarrow Q$ by establishing the logically equivalent contrapositive statement: $\neg Q \rightarrow \neg P$.
When given $P \rightarrow Q$, assume $\neg Q$ is true, then prove $\neg P$.
We prove that if the negation of the original conclusion is false, therefore the negation of the initial theorem is false.
Relies on De Morgen's Law.
Modus tollens.
<table>
<tr><th>p</th><th>q</th><th>If → Then</th><th>Technique</th></tr>
<tr><td>F</td><td>F</td><td>T</td><td>Modus Tollens</td></tr>
<tr><td>F</td><td>T</td><td>T</td><td></td></tr>
<tr><td>T</td><td>F</td><td>F</td><td></td></tr>
<tr><td>T</td><td>T</td><td>T</td><td>Modus Ponens</td></tr>
</table>
A proof by contrapositive is a special case of a proof by contradiction.
Because a contrapositive proof relies on a condition statement, it follows that mathematical theorems that don't use condition statements can't be proven using proof by contrapositive.</td></tr>
<tr><td>Contradiction</td><td>If some statement is assumed true, and a logical contradiction occurs, then the statement must be false.
Or assume that the theorem is false and then show that some logical inconsistency arises as a result of the assumption, such as $r \wedge \neg r$.
Indirect proof.
Can also be a proof by counterexample. E.g., Assume $\neg(p \rightarrow q)$, which is equivalent to $p \wedge \neg q$.</td></tr>
<tr><td>Construction</td><td>The construction of a concrete example with a property to show that something having that property exists.
AKA proof by example.</td></tr>
<tr><td>Exhaustion / By Cases</td><td>The conclusion is established by dividing it into a finite number of cases and proving each one separately.</td></tr>
<tr><td>Induction</td><td>A single "base case" is proved, and an "induction rule" is proved that establishes that any arbitrary case implies the next case.</td></tr>
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Rules of Inference with Propositions

Rule Name	Rule Logic	Example
Hypothesis	Givens. First lines of a proof.	It is raining today. You live in McKinney, Texas.
Modus Ponens	$\begin{array}{c} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$	It is raining today. If it is raining today, I will not ride my bike to school. Therefore, I will not ride my bike to school.
Modus Tollens	$\begin{array}{c} \neg q \\ \underline{p \rightarrow q} \\ \therefore \neg p \end{array}$	If Sam studied for his test, then Sam passed his test. Sam did not pass his test. Therefore, Sam did not study for his test.
Addition	$\begin{array}{c} \underline{p} \\ \therefore p \vee q \end{array}$	It is raining today. Therefore, it is either It is raining today or snowing today or both.
Simplification	$\begin{array}{c} \underline{p \wedge q} \\ \therefore p \end{array}$	It is rainy today and it is windy today. Therefore, it is rainy today.
Conjunction	$\begin{array}{c} p \\ \underline{q} \\ \therefore p \wedge q \end{array}$	Sam studied for his test. Sam passed his test. Therefore, Sam studied for his test and Sam passed his test.
Hypothetical Syllogism	$\begin{array}{c} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$	If you are mad then you will yell. If you yell then you will wake the baby. Therefore, if you are mad then you will wake the baby.
Disjunctive Syllogism	$\begin{array}{c} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$	Sam studied for his test or Sam took a nap. Sam did not study for his test. Therefore, Sam took a nap.
Resolution	$\begin{array}{c} p \vee q \\ \underline{\neg p \vee q} \\ \therefore q \vee r \end{array}$	Your shirt is red or your pants are blue. Your shirt is not red or your pants are blue. Therefore, your pants are blue or your shoes are white.
Laws of Logic	(See Harold's Logic Cheat Sheet)	

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Rules of Inference with Quantifiers

Rule Name	Rule Logic	Example
Variables	**x** : Quantified variable	The domain is the set of all integers.
Elements	**c, d** : Elements of the domain, arbitrary or particular	c is a particular integer. Element definition.
Universal Instantiation	c is an element (arbitrary or particular) $\underline{\forall x\ P(x)}$ ∴ P(c)	Sam is a student in the class. Every student in the class completed the assignment. Therefore, Sam completed his assignment.
Universal Generalization	c is an arbitrary element $\underline{P(c)\qquad}$ ∴ ∀x P(x)	Let c be an arbitrary integer. $c \le c^2$ Therefore, every integer is less than or equal to its square.
Existential Instantiation*	$\underline{\exists x\ P(x)}$ ∴ (c is a particular element) ∧ P(c)	There is an integer that is equal to its square. Therefore, $c^2 = c$, for some integer c. i.e., If an object is known to exist, then that object can be given a name.
Existential Generalization	c is an element (arbitrary or particular) $\underline{P(c)\qquad}$ ∴ ∃x P(x)	Sam is a particular student in the class. Sam completed the assignment. Therefore, there is a student in the class who completed the assignment.

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Logical Proof Example #1

Hypothesis	Proof Logic		Step Justification
Argument	If it is raining or windy or both, the game will be cancelled. The game will not be cancelled. It is not windy.		
w: *It is windy* r: *It is raining* c: *the game will bne cancelled* $(r \lor w) \to c$ $\neg c$ $\therefore \neg w$	1.	$(r \lor w) \to c$	Hypothesis
	2.	$\neg c$	Hypothesis
	3.	$\neg(r \lor w)$	Modus Tollens,1, 2
	4.	$\neg r \land \neg w$	De Morgan's Law, 3
	5.	$\neg w \land \neg r$	Commutative Law, 4
	6.	$\neg w$	Simplification, 5

Logical Proof Example #2

Hypothesis	Proof Logic		Step Justification
Argument	Every student who stayed up too late missed the test. Juan is enrolled in the class. Juan did not miss the test. ∴ Some student did not stay up too late.		
$S(x)$: *x stayed up too late* $M(x)$: *x missed the test* $\forall x\ (S(x) \to M(x))$ *Juan, a student in the class* $\neg M(Juan)$ $\therefore \exists x\ \neg S(x)$	1.	∀x (S(x) → M(x))	Hypothesis
	2.	Juan, a student in the class	Hypothesis
	3.	S(Juan) → M(Juan)	Universal Instantiation, 1, 2
	4.	¬M(Juan)	Hypothesis
	5.	¬S(Juan)	Modus Tollens, 3, 4
	6.	∃x ¬S(x)	General Instantiation, 2, 5

Proof Best Practices

#	Best Practice
1.	Indicate when the proof starts and ends. (e.g, Proof:)
2.	Write proofs in complete sentences.
3.	Justify every step of a proof using allowed assumptions.
4.	If the proof is long, give the reader a roadmap of what has been shown, what is assumed, and where the proof is going.
5.	Introduce each variable when the variable is used for the first time.
6.	A block of equations should be introduced with English text. Each step that does not follow from algebra should be justified.

Proof Language Template

<table>
<tr><th>Action</th><th colspan="2">Example Proof Statement</th></tr>
<tr><td>0. WLOG</td><td colspan="2">Without loss of generality (WLOG or w.l.o.g.), assume _____.
Can use only if doing proof by cases. (OPTIONAL)</td></tr>
<tr><td>1. Suppose</td><td>Theorem:
Suppose _____, and suppose _____.
Assume _____,</td><td>Prove:
If _____, then _____.
By hypothesis, _____.</td></tr>
<tr><td>2. Assumptions</td><td colspan="2">Proof :
Let x be an arbitrary element of _____.
Assume that x is an arbitrary real number, and suppose <equation>.
Let x be an integer, where <equation>.</td></tr>
<tr><td>3. Proof Strategy</td><td colspan="2">We shall show/prove that _____. (OPTIONAL)
We will prove the contrapositive.</td></tr>
<tr><td>4. Clarification</td><td colspan="2">By definition _____.
By assumption _____.
In other words _____.
_____ gives/yields/ to get _____.
This means that _____ and _____.
Then the definition of _____ tells us that _____.
Substituting this into the equation <equation > , we get <equations>, so <equation>.
Via algebraic manipulations, _____ and therefore _____.</td></tr>
<tr><td>5a. Since</td><td colspan="2">Since _____ and _____, by the definition of _____, _____.
Since _____, it follows that _____, and since _____ are _____, we must have _____.
Then since _____, _____, so _____.
Because we know that _____, then _____.</td></tr>
<tr><td>5b. Contradiction</td><td colspan="2">But this contradicts the fact that _____.</td></tr>
<tr><td>6. Thus</td><td>Thus, if _____ then _____.
Therefore, _____.
It follows that _____.</td><td>Then <equation>.
Hence, _____.
We can show that _____.</td></tr>
<tr><td>7. Conclusion</td><td colspan="2">Since x was an arbitrary element of _____, we can conclude that <logic>, so <set>.
But x was an arbitrary element of _____, so this shows that _____, as required.
Therefore, we have proved that _____.
Therefore, by the definition of _____ again, we have shown that _____.
… and thus it is proved.
Thus, if <equation> and <equation> then <equation>.

Thus, it cannot be the case that ___ is an element of _____ but not _____, so _____.
Since the assumption that _____ has led to a contradiction, there must be _____.</td></tr>
<tr><td>∴</td><td colspan="2">Conclusion :
Is generally used before a logical consequence, such as the conclusion of a proof.</td></tr>
<tr><td>□, ◻, ■, ▪,
Q.E.D.</td><td colspan="2">Indicates the end of a proof. This abbreviation stands for "quod erat demonstrandum", which is Latin for "that which was to be demonstrated".</td></tr>
</table>

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Direct Proof Strategies: General

Strategy	Form	Description
Write Out the Definitions	*Logical form of statement*	In many cases the logical form of a statement can be discovered by writing out the meaning or definition of some mathematical word or symbol that occurs in the statement.
Mathematical Truths		• Definitions • Theorems • Axioms • Computations
Next Steps		When analyzing the logical forms of givens and goals in order to figure out a proof, it is usually best to do only as much of the analysis as is needed to determine the <u>next step</u> of the proof. Going further with the logical analysis usually just introduces unnecessary complication, without providing any benefit.
False Starts		When trying to write a proof you may make a few false starts before finding the right way to proceed. *"We tried both methods, and the second worked."*
Nested Logic		This means that whenever you use one of these strategies you can write a sentence or two at the beginning or end of the proof and then forget about the original problem and work instead on the new problem, which will usually be easier. <u>Form:</u> Suppose $\neg R$. Suppose P. Since P and $P \rightarrow (Q \rightarrow R)$, it follows that $Q \rightarrow R$. [Proof of $\neg Q$ goes here.] Therefore $P \rightarrow \neg Q$. Therefore $\neg R \rightarrow (P \rightarrow \neg Q)$.
Reuse		Once you have shown that a statement is true, you can use it later in the proof exactly as if it were a hypothesis.
Counterexample	**goal** of the form P, try to show $\neg P$	If you find a counterexample to a theorem, then you can be sure that the theorem is incorrect.
Variables	x_0, A_0	Variables must always be introduced <u>before</u> they are used.

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Concise Write-up		A proof should contain only the reasoning needed to justify the conclusion of the proof, **not** an explanation of how you thought of that reasoning. Although we have used the symbols of logic freely in the scratch work, we have not used them in the final write-up of the proof. Stick to ordinary English. Replace → with 'then' or 'therefore'. Use of set notation is acceptable. The efficiency of exposition is one of the most attractive features of proofs, but it also often makes them difficult to read.

Proof Strategy: Transforming ($P \rightarrow Q$)

Strategy	Form	Description
Transforming		Transform the problem into one that is equivalent but easier to solve. Revise your givens and goal in some way.
To prove a **goal**	$P \rightarrow Q$	Assume P is true and then prove *Q*. If the form is $P \rightarrow Q$, then you can transform the problem by adding *P* to your list of hypotheses (givens) and changing your conclusion (goal) from $P \rightarrow Q$ to *Q*.
	$P \rightarrow Q$	Assume *P* is true and then prove *Q*. Form: Suppose *P*. [Proof of *Q* goes here.] Therefore $P \rightarrow Q$.
	$P \rightarrow Q$	Prove the contrapositive. Assume that Q is false and prove that P is false. Form: Suppose *Q* is false. [Proof of $\neg P$ goes here.] Therefore $P \rightarrow Q$.

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Proof Strategy: Inference Rules ($P \to Q$)

Strategy	Form	Description
Inference Rules		
To use a **given**	$\neg P$	If possible, reexpress this given in some other form.
	$P \to Q$	if you know that both P and $P \to Q$ are true, you can conclude that Q must also be true. (***modus ponens***)
	$P \to Q$	if you know that $P \to Q$ is true and Q is false, you can conclude that P must also be false. (***modus tollens***) aka Contrapositive.

Proof Strategy: Negations ($\neg P$)

Strategy	Form	Description
Negations	*Positive Statements*	Usually it's easier to prove a positive statement than a negative statement, so it is often helpful to reexpress a goal of the form $\neg P$ before proving it.
To prove a **goal**	$\neg P$	If possible, reexpress the goal as a positive statement, in some other form and then use one of the proof strategies for this other goal form.
	$\neg P$	Assume P is true and try to reach a **contradiction**. Once you have reached a contradiction, you can conclude that P must be false. Form: Suppose P is true. [Proof of contradiction goes here.] Thus, P is false.
To use a **given**	$\neg P$	If you're doing a proof by **contradiction**, try making P your goal. If you can prove P, then the proof will be complete, because P contradicts the given $\neg P$. Form: [Proof of P goes here.] Since we already know $\neg P$, this is a contradiction. Usually it's best to try the other strategies first if any of them apply; but if you're stuck, you can try proof by contradiction in any proof.
To prove a **goal**	$x \in B$	The goal $x \in B$ contains no logical connectives, so none of the techniques we have studied so far apply. Try reexpressing as a positive statement. Lacking anything else to do, we try proof by **contradiction**.

Proof Strategy: Quantifiers (∀*x*, ∃*x*)

<table>
<tr><th>Strategy</th><th>Form</th><th>Description</th></tr>
<tr><td>Quantifiers</td><td></td><td></td></tr>
<tr><td rowspan="2">To prove a goal</td><td>∀x P(x)</td><td>Introduce a new variable x to stand for an arbitrary object, then prove P(x).

Goal changes from ∀x P(x) to P(x).

<u>Form:</u>
Let x be arbitrary. [Element definition – arbitrary]
[Proof of P(x) goes here.]
Since x was arbitrary, we can conclude that ∀x P(x).</td></tr>
<tr><td>∃x P(x)</td><td>Try to find a value of a new variable x for which you think P(x) will be true. Then start your proof with “Let x = (the value you decided on)” and proceed to prove P(x) for this value of x.

Goal changes from ∃x P(x) to P(x) after you add a new given of x = (the value you decided on).

<u>Form:</u>
Let x = (the value you decided on). [Element definition – particular]
[Proof of P(x) goes here.]
Thus, ∃x P(x).</td></tr>
<tr><td rowspan="2">To use a given</td><td>∀x P(x)</td><td>You can plug in any value, say a, for x and use this given to conclude that P(a) is true. (universal instantiation)</td></tr>
<tr><td>∃x P(x)</td><td>If a given starts with ∃A, we should use it <u>immediately</u>.

Introduce a new variable x_0 into the proof to stand for an object for which $P(x_0)$ is true. This means that you can now assume that $P(x_0)$ is true. (existential instantiation)</td></tr>
</table>

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Proof Strategy: Existence and Uniqueness ($\exists x$, $\exists! x$)

Strategy	Form	Description
Existence & Uniqueness		
To prove a **goal**	$\exists! x\, P(x)$	Prove $\exists x\, P(x)$ and $\forall y\, \forall z\, ((P(y) \wedge P(z)) \rightarrow y = z)$. The first of these goals shows that there exists an x such that P(x) is true, and the second shows that it is unique. The two parts of the proof are therefore sometimes labeled **existence and uniqueness.** Each part is proven using strategies discussed earlier. Form: Existence: [Proof of $\exists x P(x)$ goes here.] Uniqueness: [Proof of $\forall y \forall z((P(y) \wedge P(z)) \rightarrow y = z)$ goes here.]
	$\exists! x\, P(x)$	Prove $\exists x\, (P(x) \wedge \forall y\, (P(y) \rightarrow y = x))$, using strategies.
To use a **given**	$\exists! x\, P(x)$	Treat this as two given statements, $\exists x\, P(x)$ and $\forall y\, \forall z\, ((P(y) \wedge P(z)) \rightarrow y = z)$. To use the first statement you should probably choose a name, say x_0, to stand for some object such that $P(x_0)$ is true. The second tells you that if you ever come across two objects *y* and *z* such that *P(y)* and *P(z)* are both true, you can conclude that $y = z$.

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Proof Strategy: Biconditionals ($P \leftrightarrow Q$)

Strategy	Form	Description
Biconditionals	$(P \rightarrow Q) \wedge (Q \rightarrow P)$	Can use string of equivalences. *P* iff *R* iff *Q*.
To prove a **goal**	$P \leftrightarrow Q$	Prove $P \rightarrow Q$ and $Q \rightarrow P$ separately. Once proven, it can mean equivalence, or $P = Q$. <u>Form:</u> If A then B. If B then A.
To use a **given**	$P \leftrightarrow Q$	Treat this as two separate givens: $P \rightarrow Q$, and $Q \rightarrow P$.

Proof Strategy: Conjunctions (P ∧ Q)

Strategy	Form	Description
Conjunctions		
To prove a **goal**	$P \wedge Q$	Prove P and Q separately. In other words, treat this as two separate goals: P, and Q. <u>Form:</u> Let x be arbitrary. Suppose x ϵ A. [Proof of x ∈ B goes here.] [Proof of x ∈ C goes here.] Thus, x ϵ B ∧ x ∈ C, so … Therefore … Since x was arbitrary, …
To use a **given**	$P \wedge Q$	Treat this as two separate givens: P, and Q.

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Proof Strategy: Disjunctions ($P \vee Q$)

Strategy	Form	Description
Disjunctions		
To prove a **goal**	$P \vee Q$	Break your proof into cases. In each case, either prove *P* or prove *Q*.
	$P \vee Q$	If *P* is true, then clearly the goal $P \vee Q$ is true, so you only need to worry about the case in which *P* is false. You can complete the proof in this case by proving that *Q* is true. <u>Scratch Work:</u> *Givens* *Goal* $\neg P$ *Q*
To use a **given**	$P \vee Q$	Break your proof into cases. For case 1, assume that *P* is true and use this assumption to prove the goal. For case 2, assume *Q* is true and give another proof of the goal. <u>Form:</u> Suppose _____. Let x be an arbitrary element of _____. Then either x ∈ _____ or x ∈ _____. We will consider these cases separately. Case 1. x ∈ _____. Then since _____, x ∈ _____. Case 2. x ∈ _____. Then since _____, x ∈ _____. Since we know that either x ∈ _____ or x ∈ _____, these cases cover all the possibilities, so we can conclude that x ∈ _____. Since x was an arbitrary element of _____, <this means / we can conclude> that _____.
	$P \vee Q$	If you are also given $\neg P$, or you can prove that *P* is false, then you can use this given to conclude that *Q* is true. Similarly, if you are given $\neg Q$ or can prove that *Q* is false, then you can conclude that *P* is true.

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Proof Strategy: Induction

<table>
<tr><th>Strategy</th><th>Form</th><th>Description</th></tr>
<tr><td>Induction</td><td>Natural Numbers</td><td>Usually used for proving statements about elements in a sequence.</td></tr>
<tr><td rowspan="2">To prove a goal</td><td>$\forall n \in N\ P(n)$</td><td>Mathematical Induction :
Used with natural numbers.

First prove $P(0)$ (base case).
Then prove $\forall n \in \mathbb{N}\ (P(k) \rightarrow P(k + 1))$ (induction steps).

Form k=0:
We use mathematical induction.

Base case: Setting $n = 0$, we get <Proof of $P(0)$> as required.

Induction step: Let k be an arbitrary natural number and suppose that [$P(k)$ formula or inductive hypothesis]. Then <Proof of $\forall k \in \mathbb{N}\ (P(k) \rightarrow P(k + 1))$>.

Therefore [$P(k + 1)$], as required.

Form k=1 or m:
Proof:
By induction on n.

Base case: Setting $n = 1$ or m, we get <Proof of P(1 or m)> as required.

Inductive step: We will show that for any integer $k \geq 1$ or m, if ___f(k)___, then ___f(k+1)___.
Suppose that for positive integer k, that <$P(k)$ formula>.
Then <Proof of $\forall k \in \mathbb{Z}^+\ (P(k) \rightarrow P(k + 1))$>.

Therefore <$P(k + 1)$>, as required.</td></tr>
<tr><td>$\forall n \in N\ [(\forall k < n\ P(k)) \rightarrow P(n)]$</td><td>Strong induction :
Used with sets and recursive procedures.

Form: Same Forms as above.

Example : "finite and nonempty" means that it has n elements, for some $n \in \mathbb{N}$, $n \geq 1$.

… by the inductive hypothesis [$P(k)$] …</td></tr>
</table>

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Sources:

- SNHU MAT 229 - Mathematical Proof and Problem Solving, How To Prove It - A Structured Approach, 3rd Edition - Daniel J. Vellman, Cambridge University Press, 2019.
- See also “Harold’s Logic Cheat Sheet”.

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