Harold's Partial Fractions
Cheat Sheet
15 April 2016

Partial Fractions	(http://en.wikipedia.org/wiki/Partial_fraction_decomposition)
Condition	$f(x) = \frac{P(x)}{Q(x)} = \frac{ax^n + \cdots + b}{cx^m + \cdots + d}$ where $P(x)$ *and* $Q(x)$ are polynomials
Preparation	Case 1: $n \geq m$, Peform long division first Case 2: $n < m$, Proceed to the cases below
Case I: Simple linear (1^{st} degree)	$\frac{A}{(ax+b)}$
Case II: Multiple degree linear (1^{st} degree)	$\frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}$
Case III: Simple quadratic (2^{nd} degree)	$\frac{Ax+B}{(ax^2+bx+c)}$
Case IV: Multiple degree quadratic (2^{nd} degree)	$\frac{Ax+B}{(ax^2+bx+c)} + \frac{Cx+D}{(ax^2+bx+c)^2} + \frac{Ex+F}{(ax^2+bx+c)^3}$

Example Expansion	$\frac{P(x)}{(ax+b)(cx+d)^2(ex^2+fx+g)}$ $= \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2} + \frac{Dx+E}{(ex^2+fx+g)}$

Typical Solution for Cases I & II	$\int \frac{a}{x+b}\, dx = a\, \ln\lvert x+b\rvert + C$
Typical Solution for Cases III & IV	$\int \frac{a}{x^2+b^2}\, dx = \frac{a}{b}\, tan^{-1}\left(\frac{x}{b}\right) + C$

Steps to Solve	Example
1. Write down problem	$\int \frac{5x+1}{2x^2-x-1}\, dx$
2. Check if long division is needed	Not needed since degree of numerator (top) is less than degree of denominator (bottom)
3. Factor denominator of function	$\frac{5x+1}{(2x+1)(x-1)}$
4. Expand function with A, B, Cs	$\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{(2x+1)} + \frac{B}{(x-1)}$
5. Get a common denominator	$= \frac{A(x-1)}{(2x+1)(x-1)} + \frac{B(2x+1)}{(2x+1)(x-1)}$
6. Focus on numerator	$5x+1 = A(x-1) + B(2x+1)$
7. Expand/distribute/FOIL	$5x+1 = Ax - A + 2Bx + B$

8. Separate	$5x + 1 = Ax + 2Bx - A + B$
9. Factor	$(5)x + (1) = (A + 2B)x + (-A + B)$
10. Setup system of equations	$A + 2B = 5$ $-A + B = 1$
11. Solve system of equations	
a. Substitution method	$B = A + 1$ $A + 2(A + 1) = 5$ $A + 2A + 2 = 5$ $3A = 3$ $A = 1$ $B = A + 1 = 1 + 1 = 2$ $\boldsymbol{A = 1}$ $\boldsymbol{B = 2}$
b. Elimination method	$A + 2B = 5$ $+ [-A + B = 1]$ --- $3B = 6$ $\boldsymbol{B = 2}$ $A + 2B = 5$ $-2\,[-A + B = 1]$ --- $3A = 3$ $\boldsymbol{A = 1}$
c. Matrix method	$\begin{bmatrix} A & B & k \\ A & B & k \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & 1 \end{bmatrix}$ Use TI-84 $rref()$ function $= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ $\boldsymbol{A = 1}$ $\boldsymbol{B = 2}$
12. Reassemble newly expanded function	$\frac{5x + 1}{(2x + 1)(x - 1)} = \frac{1}{(2x + 1)} + \frac{2}{(x - 1)}$
13. Restate problem with expanded function	$\int \frac{1}{2x + 1}\, dx + \int \frac{2}{x - 1}\, dx$
14. Integrate restated problem	$\int \frac{1}{2x + 1}\, dx$ $u = 2x + 1$ $du = 2\, dx$ $\frac{1}{2}\int \frac{1}{u}\, 2\, dx = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln\lvert u\rvert = \frac{1}{2}\ln\lvert 2x + 1\rvert$ $= \frac{1}{2}\ln\lvert 2x + 1\rvert + 2\ln\lvert x - 1\rvert + C$
15. Simplify	$= \ln\sqrt{\lvert 2x + 1\rvert} + \ln(x - 1)^2 + C$
16. DONE	$= \ln\left[\sqrt{\lvert 2x + 1\rvert}\,(x - 1)^2\right] + C$