Harold's Logic Cheat Sheet

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The 7 Basic Logical Symbols

Operator	Symbol	Example	English
1) Intersection	∧, **∧**, ∧, ⋀, ⋀	p ∧ q	• Conjunction • p and q • p, but q • despite the fact that p, q • although p, q • overlap
2) Union	∨, **∨**, ∨, ⋁, ⋁	p ∨ q	• Disjunction • p or q • inclusive or • both combined
3) Negation	¬, ¬	¬p	• not p
4) Conditional	→, →, →, ⟶, ⇒, ⟹	p → q	• if p then q • if p, q • q if p • p implies q • p only if q • q in case that p • p is sufficient for q • q is necessary for p
5) Biconditional	↔, ↔, **↔**, ⇔, ⟺	p ↔ q	• p iff q • p if and only if q • p is necessary and sufficient for q • if p then q, and conversely
6) Universal Quantifier	∀*x*	∀*x p(x)*	• for all • for any • for each
7) Existential Quantifier	∃*x*	∃*x p(x)*	• there exists • there is at least one
Equivalence	≡	$expression_1 \equiv expression_2$	• is identical to • is equivalent to • the two expressions always have the same truth value

- The structure of all mathematical statements can be understood using these symbols.
- All mathematical reasoning can be analyzed in terms of the proper use of these symbols.

Logical Connective Laws

Law	Union Example	Intersection Example
Identity Laws	p ∨ F ≡ p	p ∧ T ≡ p
Domination or Null Laws	p ∨ T ≡ T	p ∧ F ≡ F
Idempotent Laws	p ∨ p ≡ p	p ∧ p ≡ p
Double Negations or Involution Law	¬ ¬p ≡ p	
Complement or Complementary Laws	p ∨ ¬p ≡ T ¬F ≡ T	p ∧ ¬p ≡ F ¬T ≡ F
Commutative Laws	p ∨ q ≡ q ∨ p	p ∧ q ≡ q ∧ p
Associative Laws	(p ∨ q) ∨ r ≡ p ∨ (q ∨ r)	(p ∧ q) ∧ r ≡ p ∧ (q ∧ r)
Distributive Laws	p ∧ (q ∨ r) ≡ (p ∧ q) ∨ (p ∧ r)	p ∨ (q ∧ r) ≡ (p ∨ q) ∧ (p ∨ r)
Uniting Laws	(p ∧ q) ∨ (p ∧ ¬q) ≡ p	(p ∨ q) ∧ (p ∨ ¬q) ≡ p
Absorption Laws	p ∨ (p ∧ q) ≡ p	p ∧ (p ∨ q) ≡ p
De Morgan's Law (Propositional Logic)	p ∨ q ≡ ¬(¬p ∧ ¬q) ¬(p ∨ q) ≡ ¬p ∧ ¬q (p ∨ ¬q) → r ≡ ¬r → (¬p ∧ q)	p ∧ q ≡ ¬(¬p ∨ ¬q) ¬(p ∧ q) ≡ ¬p ∨ ¬q
Multiplying and Factoring Laws	(p ∨ q) ∧ (¬p ∨ r) ≡ (p ∧ r) ∨ (¬p ∧ q)	(p ∧ q) ∨ (¬p ∧ r) ≡ (p ∨ r) ∧ (¬p ∨ q)
Consensus Laws	(p ∧ q) ∨ (q ∧ r) ∨ (¬p ∧ r) ≡ (p ∧ q) ∨ (¬p ∧ r)	(p ∨ q) ∧ (q ∨ r) ∧ (¬p ∨ r) ≡ (p ∨ q) ∧ (¬p ∨ r)
Tautology Laws (⊤)	p ∨ (⊤) ≡ ⊤ p ∨ ¬p ≡ ⊤ (True)	p ∧ (⊤) ≡ p
	¬(⊤) = ⊥	
Contradiction Laws (⊥)	p ∨ (⊥) ≡ p	p ∧ (⊥) ≡ ⊥ p ∧ ¬p ≡ ⊥ (False)
	¬(⊥) ≡ ⊤	

Logical Conditional Connective Laws

Law or Statement	Logical Expression	Is Equivalent To (≡)	Description
Conditional Laws	$p \longrightarrow q$	$\neg p \lor q$ $\neg(p \land \neg q)$	Conditional, If ... Then, Implication
Biconditional Laws	$p \longleftrightarrow q$	$(p \longrightarrow q) \land (q \longrightarrow p)$ $(p \longrightarrow q) \land (\neg p \longrightarrow \neg q)$ $(p \land q) \lor (\neg p \land \neg q)$	Bi-conditional, If and only If, iff, XNOR
Sufficient Condition	p is a sufficient condition for q	The truth of p suffices to guarantee the truth of q.	
Necessary Condition	q is a necessary condition for p	In order for p to be true, it is necessary for q to be true also. $\neg q \longrightarrow \neg p$	
Equivalence	$p \longleftrightarrow q$	$p \equiv q$ $p \Longrightarrow q$	Is logically equivalent to ($p \equiv \neg\neg p$) Is equivalent to
Contrapositive	$p \longrightarrow q$	$\equiv \neg q \longrightarrow \neg p$	True
Converse*	$p \longrightarrow q$	$\not\equiv q \longrightarrow p$	False
Inverse*	$p \longrightarrow q$	$\not\equiv \neg p \longrightarrow \neg q$	False

Logical Predicates

Definition	Logical Expression	Is Equivalent To (≡)	Description
Universe of Discourse	***U***	All possible inputs in a given range	• Universe of Discourse • Universal Set • Universe
Domain of Discourse	$\mathbb{D}$	All possible inputs in a given range	• Domain of Discourse • Universe of Discourse
Proposition or Logical Statement	*p: "Roxy is a mammal"*	*p*	• Must be True or False • Cannot be a question • Cannot be a command
Predicate	*P(x): "x is a mammal"*	*P(x)*	• A logical statement whose truth value is a function of one or more <u>variables</u> • Truth depends upon the input variable *x* • P(x) ≠ a number • P(5) is a proposition
Example Statements	*q*: $\forall x \in \mathbb{D}$, *P(x)*: "x is a mammal"	*"For all x in the domain of discourse, P(x) is a mammal."*	• Is either True or False • A quantified predicate turns it into a logical statement
	T(x, y)	"*x* is a twin of *y*."	Predicate with two input variables
Truth Set (Single Free Variable)	*T = P(x)*	$T = \{a \mid P(a)\}$ $T = \{a \in A \mid P(a)\}$ $a \in T$	The set of all values of x that make the statement *p(x)* true
	Example:	$P(x_1)$, $P(x_2)$, and $P(x_3)$ are True	
Truth Set (Ordered Pair)	*T = P(x, y)*	$\{(a, b) \in A \times B \mid P(a, b)\}$ $(a, b) \in T$	Cross product truth set
	Examples:	$\{(p, n) \in P \times \mathbb{N} \mid$ the person p has n children$\} = \{$(John, 2), …$\}$ $\{(p, c, n) \in P \times C \times \mathbb{N} \mid$ the person p has lived in the city c for n years$\}$	

Logical Quantifiers

Definition	Logical Expression	Is Equivalent To (≡)	English
Universal Quantifier	$\forall x\, P(x)$ $\forall x \in P(x)$ $\forall x \in \mathbb{D}, P(x)$ $\forall x$, *if x is in* $\mathbb{D}$ *then* $P(x)$	“For all *x* in the domain, *P(x)* is true” $\forall x \in A\, P(x) \equiv \forall x\, (x \in A \longrightarrow P(x))$ For the finite set domain of discourse $\{a_1, a_2, \ldots, a_k\}$, $\forall x\, P(x) \equiv P(a_1) \wedge P(a_2) \wedge \ldots \wedge P(a_k)$	• for all • all elements • for each member • any • every • everyone • everybody • everything • x could be anything at all
Existential Quantifier	$\exists x\, P(x)$ $\exists x \in P(x)$ $\exists x \in \mathbb{D}, P(x)$	“There exists x in the domain, such that *P(x)* is true” For the finite set domain of discourse $\{a_1, a_2, \ldots, a_k\}$, $\exists x\, P(x) \equiv P(a_1) \vee P(a_2) \vee \ldots \vee P(a_k)$	• there exists an x • there is • some • someone • somebody • at least one value of x • there is at least one x • it is the case that • the truth set is not equal to ∅
Uniqueness Quantifier	$\exists! x\, P(x)$	there is a unique x in *P(x)* such that … $\exists x\, (P(x) \wedge \neg\, y\, (P(y) \wedge y \neq x))$ $\exists x\, (P(x) \wedge \forall y\, (P(y) \rightarrow y = x))$ $\exists x\, \forall y\, (P(y) \leftrightarrow y = x)$ $\exists x\, P(x) \wedge \forall y\, \forall z((P(y) \wedge P(z)) \rightarrow y = z)$	• unique • there is a unique x • there exists exactly one • there is exactly one x such that *P(x)*
Negated Existential Quantifier	$\neg\, [\exists x\, P(x)]$	$\forall x\, \neg P(x)$	• nobody • no one • not one • there does not exist
	$\neg\, [\forall x\, P(x)]$	$\exists x\, \neg P(x)$	
Order of Precedence	PEMDAS for Logic : 1. Parenthesis () 2. Logical NOT (¬) 3. Quantifiers (∀, ∃) 4. Logical AND (∧) 5. Logical OR (∨) 6. Logical Conditional (→) 7. Logical Biconditional (↔)		Applied Left to Right Example : $\forall x\, P(x) \wedge Q(x) \equiv$ $(\forall x\, P(x)) \wedge Q(x)$

Quantifier Laws

Definition	Logical Expression	Is Equivalent To (≡)	Description / Example / • English
Abbreviation	$\exists x\ (x \in A \land \neg P(x))$	$\exists x \in A\ \neg P(x)$	Simplification
Expanding Abbreviation	$\forall x \in A\ P(x)$	$\forall x\ (x \in A \to P(x))$	Complication
Quantifier Negation Laws	$\forall x\ \neg P(x)$	$\neg \exists x\ P(x)$	• nobody's perfect
	$\neg \forall x\ P(x)$	$\exists x\ \neg P(x)$	• not everyone is perfect • someone is imperfect
Conditional Law	$x \in A \longrightarrow P(x)$	$x \notin A \lor P(x)$	$p \longrightarrow q \equiv \neg p \lor q$
Subset Negation Law	$x \in A$	$\neg(x \notin A)$	Swap ∈ with ∉, or vice versa
De Morgan's Law (Quantifier Negation)	$\neg \forall x\ P(x) \equiv \exists x\ \neg P(x)$ $\neg \exists x\ P(x) \equiv \forall x\ \neg P(x)$ $\neg \forall x\ \forall y\ P(x, y) \equiv \exists x\ \exists y\ \neg P(x, y)$ $\neg \forall x\ \exists y\ P(x, y) \equiv \exists x\ \forall y\ \neg P(x, y)$ $\neg \exists x\ \forall y\ P(x, y) \equiv \forall x\ \exists y\ \neg P(x, y)$ $\neg \exists x\ \exists y\ P(x, y) \equiv \forall x\ \forall y\ \neg P(x, y)$		De Morgan's Law for single and nested quantifiers
Nested / Multiple-Quantified Statements	$\forall x\ \forall y$	$\forall y\ \forall x$	• for all objects x and y, …
	$\exists x\ \exists y$	$\exists y\ \exists x$	• there are objects x and y such that …
	$\forall x\ \exists y\ P(x, y) \not\equiv \exists x\ \forall y\ P(x, y)$		False Counterexample for $x, y \in \mathbb{Z}$: $\forall x\ \exists y\ (x + y = 0) \equiv$ True $\exists x\ \forall y\ (x + y = 0) \equiv$ False
	$\neg(\forall x\ \exists y\ P(x, y))$	$\exists x\ \forall y\ \neg P(x, y)$	Negation of multiply-quantified statements
	$\neg(\exists x\ \forall y\ P(x, y))$	$\forall x\ \exists y\ \neg P(x, y)$	
Moving Quantifiers	$\forall x\ (P(x) \to \exists y\ Q(x, y)) \equiv$ $\forall x\ \exists y\ (P(x) \to Q(x, y))$		You can move a quantifier left if variable is not used yet

Quantifier Logic Examples

Action	Logical Statement	English
Everyone	$\forall x\ \forall y\ P(x, y)$ NOTE: includes (x = y)	• everyone <did something> to everyone
Everyone Else	$\forall x\ \forall y\ (x \neq y) \to P(x, y)$ NOTE: excludes (x = y)	• everyone <did something> to everyone else
Someone Else	$\forall x\ \exists y\ ((x \neq y) \land P(x, y))$ NOTE: excludes (x = y)	• everyone <did something> to someone else
Exactly One	$\exists x\ (P(x) \land \forall y\ ((x \neq y) \to \neg P(y))) \equiv$ $\exists! x\ P(x)$	• exactly one person <did something>
No One	$\neg \exists x\ (P(x)$	• no one <did something>

Valid Quantifier Formulas

A		B
∀x (*P(x)* ∧ *Q(x)*)	≡	(∀x *P(x)* ∧ ∀x *Q(x)*)
∃x (*P(x)* ∧ *Q(x)*)	→	(∃x *P(x)* ∧ ∃x *Q(x)*)
∀x (*P(x)* ∨ *Q(x)*)	←	(∀x *P(x)* ∨ ∀x *Q(x)*)
∃x (*P(x)* ∨ *Q(x)*)	≡	(∃x *P(x)* ∨ ∃x *Q(x)*)
∀x (*P(x)* → *Q(x)*)	←	(∃x *P(x)* → ∀x *Q(x)*)
∃x (*P(x)* → *Q(x)*)	≡	(∀x *P(x)* → ∃x *Q(x)*)
∀x ¬*P(x)*	≡	¬∃x *P(x)*
∃x ¬*P(x)*	≡	¬∀x *P(x)*
∀x ∃y *T(x, y)*	←	∃y ∀x *T(x, y)*
∀x ∀y *T(x, y)*	≡	∀y ∀x *T(x, y)*
∃x ∃y *T(x, y)*	≡	∃y ∃x *T(x, y)*
∀x (*P(x)* ∨ *R*)	≡	(∀x *P(x)* ∨ *R*)
∃x (*P(x)* ∧ *R*)	≡	(∃x *P(x)* ∧ *R*)
∀x (*P(x)* → *R*)	≡	(∃x *P(x)* → *R*)
∃x (*P(x)* → *R*)	→	(∀x *P(x)* → *R*)
∀x (*R* → *Q(x)*)	≡	(R → ∀x *Q(x)*)
∃x (*R* → *Q(x)*)	→	(R → ∃x *Q(x)*)
∀x *R*	←	*R*
∃x *R*	→	*R*

Note: The above formulas are valid in classical first-order logic assuming that *x* does not occur free in *R*.

Invalid Quantifier Formulas

A		B	Counterexample
∃x (*P(x)* ∧ *Q(x)*)	←	(∃x *P(x)* ∧ ∃x *Q(x)*)	D = {*a*, *b*}, M = {*P(a)*, *Q(b)*}
∀x (*P(x)* ∨ *Q(x)*)	→	(∀x *P(x)* ∨ ∀x *Q(x)*)	D = {*a*, *b*}, M = {*P(a)*, *Q(b)*}
∀x (*P(x)* → *Q(x)*)	→	(∃x *P(x)* → ∀x *Q(x)*)	D = {*a*, *b*}, M = {*P(a)*, *Q(a)*}
∀x ∃y *T(x, y)*	→	∃y ∀x *T(x, y)*	D = {*a*, *b*}, M = {*T(a, b)*, *T(b, a)*}
∃x (*P(x)* → *R*)	←	(∀x *P(x)* → *R*)	D = Ø, M = {*R*}
∃x (*R* → *Q(x)*)	←	(*R* → ∃x *Q(x)*)	D = Ø, M = Ø
∀x *R*	→	*R*	D = Ø, M = Ø
∃x *R*	←	*R*	D = Ø, M = {*R*}

Note: if empty domains are not allowed, then the last four implications above are in fact valid.

Sources:

- SNHU MAT 230 - Discrete Mathematics, zyBooks.
- See also "Harold's Proofs Cheat Sheet".
- https://byjus.com/maths/set-theory-symbols/
- https://en.wikipedia.org/wiki/List_of_logic_symbols
- https://nokyotsu.com/qscripts/2014/07/distribution-of-quantifiers-over-logic-connectives.html

Logical Truth Tables

p	q	Conjuntion (and) ∧	NAND ⊼	Disjuntion (or) ∨	NOR ⊽	XOR ⊻, ⊕	XNOR ⊙	Negation (not) ¬P
F	**F**	F	T	F	T	F	T	
F	**T**	F	T	T	F	T	F	T
T	**F**	F	T	T	F	T	F	F
T	**T**	T	F	T	F	F	T	

p	q	Material Implication (If … Then) →	Biconditional (Iff) ↔	Tautology (True) ⊤	Contradiction (False) ⊥
F	**F**	T	T	T	F
F	**T**	T	F	T	F
T	**F**	F	F	T	F
T	**T**	T	T	T	F

Blank Truth Tables

Inputs				Output		
p	**q**	**r**	**s**	**x**	**y**	**z**
F	**F**	**F**	**F**			
F	**F**	**F**	**T**			
F	**F**	**T**	**F**			
F	**F**	**T**	**T**			
F	**T**	**F**	**F**			
F	**T**	**F**	**T**			
F	**T**	**T**	**F**			
F	**T**	**T**	**T**			
T	**F**	**F**	**F**			
T	**F**	**F**	**T**			
T	**F**	**T**	**F**			
T	**F**	**T**	**T**			
T	**T**	**F**	**F**			
T	**T**	**F**	**T**			
T	**T**	**T**	**F**			
T	**T**	**T**	**T**			

Inputs			Output	
p	**q**	**r**	**x**	**y**
F	**F**	**F**		
F	**F**	**T**		
F	**T**	**F**		
F	**T**	**T**		
T	**F**	**F**		
T	**F**	**T**		
T	**T**	**F**		
T	**T**	**T**		

Inputs		Output
p	**q**	**x**
F	**F**	
F	**T**	
T	**F**	
T	**T**	