Harold's Linear Optimization Cheat Sheet

17 October 2021

How to Optimize using the Simplex Method	

Steps	1. Read the word problem at least 4 times
	2. Assign non-basic variables (x_1, x_2, …)
	3. List optimization function, z = _____, that will be maximized
	4. List inequalities (constraints)
	5. Add basic variables, also called slack variables, (s_1, s_2, …), to turn inequalities into equations
	  a. ≤ means s_n is positive (default)
	  b. ≥ means s_n is negative
	  c. Column has all zeros (0) except for one (1) 1 for the slack variable
	6. Organize the equation and inequalities into a matrix, with variables for the columns
	7. Construct a simplex tableau corresponding to the system
	  a. Rows 1-n are the inequalities
	  b. Last row (**indicator row**) is the z equation solved to equal zero (0)
	  i. Example: if $z = 5x_1 + 7x_2$, then $-5x_1 -7x_2 + z = 0$, or -5 -7 1 | 0
	8. If the indicator row coefficients are all positive, then the problem is solved, otherwise …
	9. Find pivot
	  a. Pivot Column is the most <u>negative</u> value in indicator row on bottom
	  b. Pivot Row is the smallest <u>positive</u> ratio of pivot column coefficient to b value on far right
	10. Pivot (perform matrix row operations) to create a new simplex tableau
	  a. Example: $R_1 = R_1 – 2R_2$
	  b. All values in column should be turned into zeros (0) except the pivot element
	  c. Pivot element should be turned into one (1) using division <u>afterwards</u>
	11. Repeat steps 8 - 10 until no more negatives in the indicator row on bottom
	12. Maximum objective function value is in the simplex tableau's bottom right corner

Example	Objective Function: $z = x_1 + 2x_2 – x_3$ Subject To: $2x_1 + x_2 + x_3 \le 14$ $4x_1 + 2x_2 + 3x_3 \le 28$ $2x_1 + 5x_2 + 5x_3 \le 30$ $x_1 \ge 0; x_2 \ge 0; x_3 \ge 0$

Simplex Tableau

Adding slack variables gives:

$2x_1 + x_2 + x_3 + s_1 = 14$

$4x_1 + 2x_2 + 3x_3 + s_2 = 28$

$2x_1 + 5x_2 + 5x_3 + s_3 = 30$

where all variables $x_n \geq 0$ (e.g., not negative)

Simplex Tableau:

$$\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & b \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 14 \\ 4 & 2 & 3 & 0 & 1 & 0 & 0 & 28 \\ 2 & 5 & 5 & 0 & 0 & 1 & 0 & 30 \\ - & - & - & - & - & - & - & - \\ -1 & -2 & +1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Pivot Determination:

The -2 is the most negative on the bottom row, so pivot column is 2.

Ratios are row 1: 14/1 = 14, row 2: 28/2 = 14, row 3: 30/5 = 6.

So, the pivot is at column 2, row 3 = 5.

Row Operations:

Pivot is Col 2, Row 3.

$R_1 = R_1 - R_3$

$R_2 = R_2 - 2\,R_3$

$R_4 = R_4 + 2\,R_3$

$R_3 = (1/5)\,R_3$

Next Pivot is Col 1, Row 2.

$R_1 = R_1 - (8/5)\,R_2$

$R_3 = R_3 - (2/5)\,R_2$

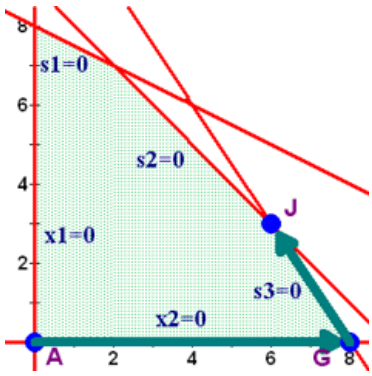
$R_4 = R_4 + (1/5)\,R_2$

$R_2 = (5/16)\,R_2$

Final Tableau:

$$\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & b \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & \frac{5}{16} & 0 & \frac{5}{16} & -\frac{1}{8} & 0 & 5 \\ 0 & 1 & \frac{7}{8} & 0 & -\frac{1}{8} & \frac{1}{4} & 0 & 4 \\ - & - & - & - & - & - & - & - \\ 0 & 0 & \frac{49}{16} & 0 & \frac{1}{16} & \frac{3}{8} & 1 & 13 \end{array}\right]$$

Note: All indicators in bottom row are now zero or larger. 13 is not an indicator.

Basic Feasible Solution	$x_1 = 5$ $x_2 = 4$ $x_3 = 0$ $s_1 = 0$ $s_2 = 0$ $s_3 = 0$ $z = 13$ Since all slack variables $s_n \geq 0$, this solution is optimal.	Choose 5 x_1s Choose 4 x_2s Choose no x_3s Objective function value of 13.	

Sources: https://math.uww.edu/~mcfarlat/s-prob.htm