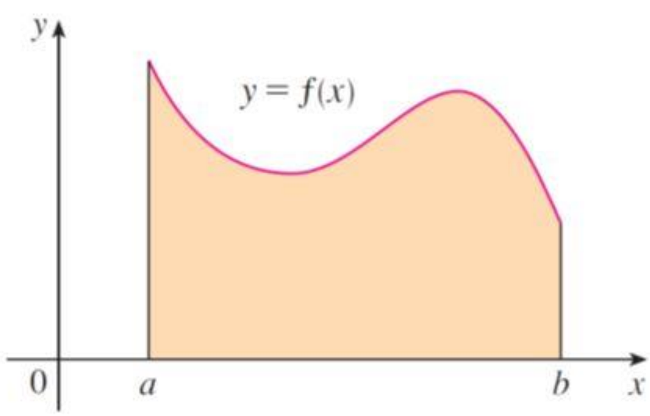
Harold's Fundamental Theorems of Calculus
"Cheat Sheet"

17 October 2020

Formulas	Examples
1) The First Fundamental Theorem of Calculus: Integrating Derivatives $\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(x)\|_a^b = F(b) - F(a)$	1) Solve $\int_0^4 e^x\,dx$ $f(x) = F'(x) = e^x$ $F(x) = e^x$ $F(b) - F(a) = e^4 - e^0 = \mathbf{e^4 - 1}$
2) The Second Fundamental Theorem of Calculus: Differentiating Integrals Simple Formula $\frac{d}{dx}\int_a^x f(t)\,dt = F'(x) = f(x)$ General Formula $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f\big(h(x)\big)h'(x) - f\big(g(x)\big)g'(x)$ 	2) Solve $\frac{d}{dx}\int_{4x}^{x^2} e^t\,dt$ 2nd FToC General Formula $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f\big(h(x)\big)\,h'(x) - f\big(g(x)\big)\,g'(x)$ Determine functions $f(t) = e^t$ $g(x) = 4x$ $h(x) = x^2$ Substitute $f\big(h(x)\big) = e^{x^2}$ $f\big(g(x)\big) = e^{4x}$ Differentiate $h'(x) = \frac{d}{dx}x^2 = 2x$ $g'(x) = \frac{d}{dx}4x = 4$ Plug them back into the formula $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f\big(h(x)\big)\,h'(x) - f\big(g(x)\big)\,g'(x)$ $= e^{x^2}(2x) - e^{4x}(4)$ Simplify $= \mathbf{2xe^{x^2} - 4e^{4x}}$

Second FTC Proof #1	Second FTC Proof #2

Second FTC Proof #1

Proof of the Second Fundamental Theorem of Calculus :

a) Break integral into two parts

$$\int_{g(x)}^{h(x)} f(t)\,dt = \int_{g(x)}^{a} f(t)\,dt + \int_{a}^{h(x)} f(t)\,dt$$

b) Change bounds

$$\int_{g(x)}^{h(x)} f(t)\,dt = -\int_{a}^{g(x)} f(t)\,dt + \int_{a}^{h(x)} f(t)\,dt$$

$$= \int_{a}^{h(x)} f(t)\,dt - \int_{a}^{g(x)} f(t)\,dt$$

c) Apply the First Fundamantal Theorem of Calculus

$$\int_{g(x)}^{h(x)} f(t)\,dt = F\big(h(x)\big) - F\big(g(x)\big)$$

d) Take the derivative

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = \frac{d}{dx}\,F\big(h(x)\big) - \frac{d}{dx}\,F(g(x))$$

e) Use the derivative chain rule

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = F'\big(h(x)\big)h'(x) - F'\big(g(x)\big)g'(x)$$

f) Using the Second Fundamantal Theorem of Calculus we see that $F'\big(h(x)\big) = f\big(h(x)\big)$ and $F'\big(g(x)\big) = f\big(g(x)\big)$.

g) This completes the proof of the general formula.

Second FTC Proof #2

Proof of the Second Fundamental Theorem of Calculus :

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = \frac{d}{dx}[F(t)]_{g(x)}^{h(x)}$$

a) Apply the First Fundamantal Theorem of Calculus

$$= \frac{d}{dx}\big[F\big(h(x)\big) - F\big(g(x)\big)\big]$$

b) Distribute

$$= \frac{d}{dx}F\big(h(x)\big) - \frac{d}{dx}F\big(g(x)\big)$$

c) Chain rule

$$= F'\big(h(x)\big)\,\frac{d}{dx}h(x) - F'\big(g(x)\big)\,\frac{d}{dx}g(x)$$

d) Simplify

$$= F'\big(h(x)\big)\,h'(x) - F'\big(g(x)\big)\,g'(x)$$

e) Substitute

$$= f\big(h(x)\big)\,h'(x) - f\big(g(x)\big)\,g'(x)$$

Formulas used in the proofs:

Equivalent Notation

$$y = f(x)$$

$$y' = \frac{dy}{dx} = f'(x) = \frac{df(x)}{dx}$$

Change of Bounds

$$\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx$$

Derivative Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big)\,g'(x)$$