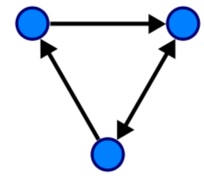
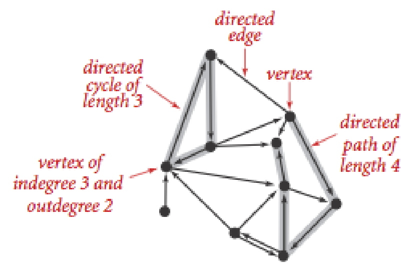
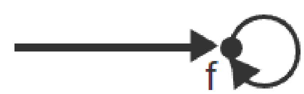
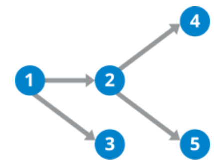
Harold's Directed Graphs
Cheat Sheet

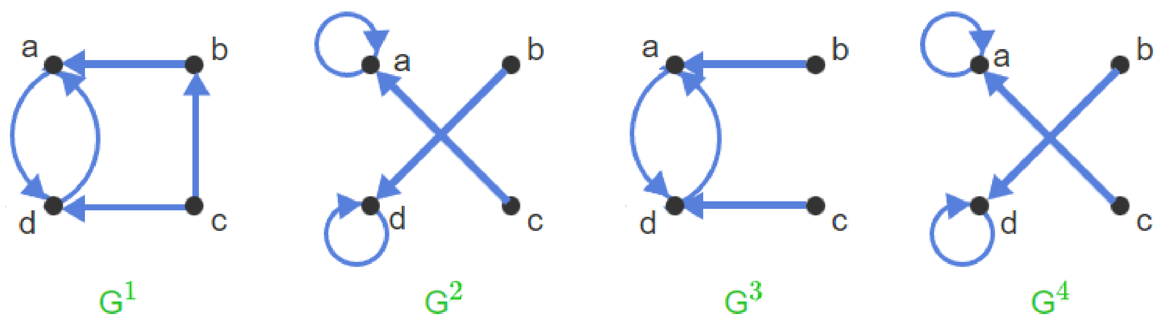
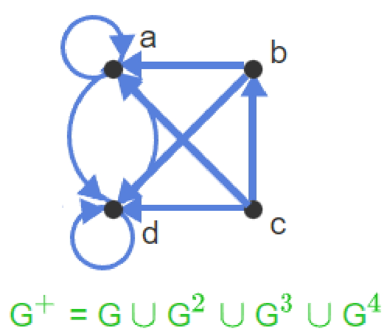
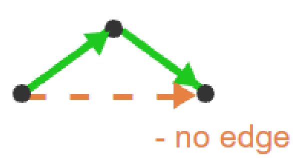
22 October 2022

Definitions

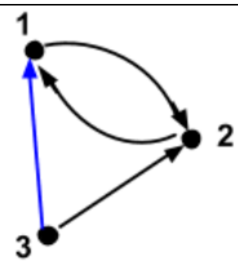
Term	Definition	Example
Vertices (Nodes)	An individual element of V is called a <u>vertex</u>.	Set $V = \{a, b, c, d, e\}$ ① or ●
Edges (Arcs)	A directed <u>edge</u> (u, v) ∈ E, is pictured as an arrow going from one vertex to another.	Set E ⊆ V x V $E = \{(a, b), (a, c), \ldots, (d, e)\}$ 
Directed Graph (Digraph)	A finite set of dots called <u>vertices</u> (or <u>nodes</u>) that are connected by links called <u>edges</u> (or <u>arcs</u>). Consists of a pair (V, E). A sequence of vertices in which there is a (directed) edge pointing from each vertex in the sequence to its successor in the sequence, with no repeated edges.	 Anatomy of a digraph
Self-Loop (Loop)	An edge that connects a vertex to itself.	
In-Degree	The number of edges pointing into, to, or with v as their terminal vertex.	$in - degree(v) =$ $\lvert\{\, u \mid (u, v) \in E \,\}\rvert$
Out-Degree	The number of edges pointing out of, from, or with v as their initial vertex.	$out - degree(v) =$ $\lvert\{\, u \mid (v, u) \in E \,\}\rvert$
Walk	A sequence of alternating vertices and edges that starts and ends with a vertex.	$\langle v_0, (v_0, v_1), v_1, (v_1, v_2), v_2, \ldots, v_l$ $\langle v_0, v_1, v_2, \ldots, v_l \rangle$
Open Walk	A walk in which the first and last vertices are not the same.	$\langle a, \ldots, z \rangle$
Closed Walk	A walk in which the first and last vertices are the same.	$\langle a, \ldots, a \rangle$
Length	l, the number of edges in the walk, path, or cycle.	$I = \lvert E \rvert$
Trail	An <u>open</u> walk in which no <u>edge</u> occurs more than once.	$\langle a, b, c, d, c, b, a \rangle$
Circuit	A <u>closed</u> walk in which no <u>edge</u> occurs more than once.	$\langle a, b, a, c, a \rangle$

Path	A trail in which no vertex occurs more than once.	$\langle a, b, c, d \rangle$
Cycle	A circuit of length at least 1 in which no vertex occurs more than once, except the first and last vertices which are the same.	$\langle a, b, c, a \rangle$
DAG	A directed acyclic graph (or DAG) is a digraph with no directed cycles.	

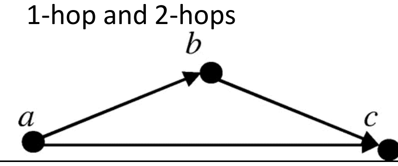
Digraph Theorems

<table>
<tr><th>Theorem</th><th>Definition and Examples</th></tr>
<tr><td>Graph Power Theorem (G^k)</td><td>Let G be a directed graph.
Let u and v be any two vertices in G.
There is an edge from u to v in G^k if and only if there is a walk of length k from u to v in G.</td></tr>
<tr><td colspan="2">
</td></tr>
<tr><td rowspan="2">Transitive Closure</td><td>The union of G^k for all $k \geq 1$ (denoted G^+) represents <u>reachability</u> by walks of any length in G.
$G^+ = G^1 \cup G^2 \cup G^3 \cup G^4 \ldots$ (infinite or up to $|V|$)
$G^+ = G^1 \cup G^2 \cup G^3 \cup \ldots \cup G^n$ (finite with n vertices)
$R^+ = R^1 \cup R^2 \cup R^3 \cup \ldots \cup R^n$ (finite with n elements)</td></tr>
<tr><td>
</td></tr>
<tr><td rowspan="2">Procedure to find the transitive closure of a relation R on a set A</td><td>Repeat the following step until no pair is added to R:
● If there are three elements x, y, z ∈ A such that (x, y) ∈ R, (y, z) ∈ R and (x, z) ∉ R, then add (x, z) to R.</td></tr>
<tr><td>
</td></tr>
</table>

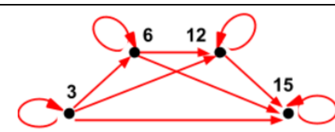
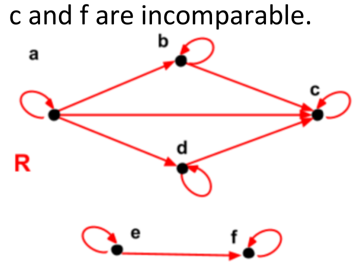
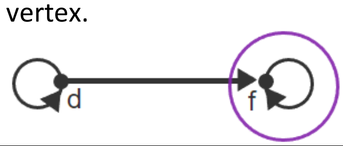
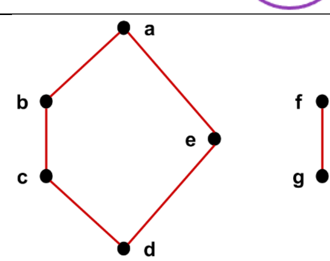
Boolean Matrix Operations

Term	Description
Adjacency Matrix	A directed graph G with n vertices that is represented by an n × n matrix over the set {0, 1}. $A_{i,j} = 1$ if there is an edge from vertex i to vertex j in G, otherwise, $A_{i,j} = 0$.
	 $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$
Boolean Matrix **{0, 1}**	A matrix whose entries are from the set {0, 1}. **Purpose:** Matrix addition and multiplication for square Boolean matrices are used to compute the transitive closure of a graph.
Dot Product	For Boolean matrices, if the dot product (sum of products) ≥ 1, then dot product = 1.
Matrix Product **(AB)**	The product of two matrices, A and B, is **well defined** only if the number of columns in A is equal to the number of rows in B. Associative, but not commutative.
k^{th} Power of a Matrix **(A^k)**	$A^2 = A \bullet A$ $A^3 = A^2 \bullet A$
Matrix A^k is the Adjacency Matrix for Graph G^k	Let G be a directed graph with n vertices and let A be the adjacency matrix for G. Then for any k ≥ 1, A^k is the adjacency matrix of G^k, where Boolean addition and multiplication are used to compute A^k. There is a walk of length k in G from vertex v to vertex w if and only if the entry in row v, column w in A^k is 1.
	How to read it: There is a walk of length 3 in G from vertex 1 to vertex 3 if and only if there is an edge from 1 to 3 in G^3. If row 1, column 3 of A^3 is 0, then no such walk exists.
Matrix Sum **(A+B)**	The sum of two matrices A and B is **well defined** if A and B have the same number of rows and the same number of columns.
	For Boolean matrices, if the sum ≥ 1, then sum = 1.
Addition and Graph Union	Let G and H be two directed graphs with the same vertex set. Let A be the adjacency matrix for G and B the adjacency matrix for H. Then the adjacency matrix for **G U H = A + B**, where Boolean addition is used on the entries of matrices A and B.
Transitive Closure of G^+	Includes both Boolean multiplication and addition. $G^+ = G^1 \cup G^2 \cup G^3 \cup \ldots \cup G^n$ $A^+ = A^1 \cup A^2 \cup A^3 \cup \ldots \cup A^n$ A^+ shows every possible walk in G^+ up to length n.

Order Properties of Binary Relations with Two Sets

Property	Logical Statement	Description
Reflexive	xRx $(x, x) \in R$ $\forall x \in A\ (xRx)$ $\forall x \in A\ ((x, x) \in R)$	• $i_A \subseteq R$ where i_A is the identity relation of set A or $i_A = \{(x, x) \mid x \in A\}$ • **Directed graph**: Loop 
Anti-Reflexive	$\neg (xRx)$ $\forall x \in A\ \neg (xRx)$	• **Directed graph**: No loops
Symmetric	$xRy \longrightarrow yRx$ $\forall x \in A\ \forall y \in A\ (xRy \longrightarrow yRx)$	• $R = R^{-1}$ • **Directed graph**: 2-way arrow (edges come in pairs) or no arrows
Anti-Symmetric	$(xRy \land yRx) \longrightarrow (x = y)$ $(x \neq y) \longrightarrow \neg (xRy) \lor \neg (yRx)$ $\forall x \in A\ \forall y \in A\ ((xRy \land yRx) \longrightarrow (x = y))$	• Equivalence • **Directed graph**: An arrow from x to y implies that there is no arrow from y to x No: 
Asymmetric	$xRy \longrightarrow \neg (yRx)$ $\forall x \in A\ \forall y \in A\ \forall z \in A\ (xRy \longrightarrow \neg (yRx))$	• Fails the vertical line test, so not a proper function, f(x) • **Directed graph**: 1-way arrow
Transitive	$(xRy \land yRz) \longrightarrow xRz$ $\forall x\ \forall y\ \forall z\ ((xRy \land yRz) \longrightarrow xRz)$ $\forall x \in A\ \forall y \in A\ \forall z \in A\ ((xRy \land yRz) \longrightarrow xRz)$	• $R \circ R \subseteq R$ • Similar to $S \circ R$ • **Directed graph**: Two routes from every vertex A to every vertex B, 1-hop and 2-hops 
Total	$xRy \lor yRx$ $\forall x \in A\ \forall y \in A\ (xRy \lor yRx)$	• Either-or
Density	$xRy \longrightarrow \exists z \mid xRz \land zRy$ $\forall x \in A\ \forall y\ (xRy) \longrightarrow \exists z \mid xRz \land zRy$	• A middle-man exists
Binary	$R^{-1} \circ R$ = Relation on set A $R \circ R^{-1}$ = Relation on set C	• Relation on set <set> • Binary relation on set <set>
Identity	$i_A = \{(x, y) \in A \times A \mid x = y\}$ $i_A = \{(x, x) \mid x \in A\}$	• Similar to a diagonal matrix
Composition ($S \circ R$)	$S \circ R = (a, c) \in S \circ R \longleftrightarrow \exists b \mid (a, b) \in R$ and $(b, c) \in S$ $\{(a, c) \in A \times C \mid \exists b \in B\ ((a, b) \in R$ and $(b, c) \in S)\}$ aRb and bSc $\{(a, c) \in A \times C \mid \exists b \in B\ (aRb \land bSc)\}$	• The composition of S and R is the relation $S \circ R$ from A to C • aRb and bSc, meaning R:a ⟶ R:b ⟶ S:b ⟶ S:c, so (R:a, S:c) • Ring operator

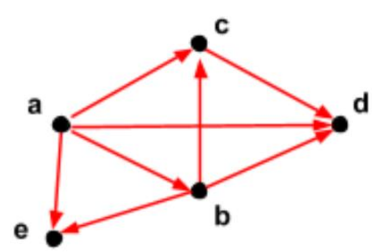
Partial Orders

Term	Description	Additional
Partial Order	A relation R on a set A that is **reflexive**, **transitive**, and **anti-symmetric**. A partial order acts like a ≤ operator on the elements of A.	aRb = a ⪯ b "a is at most b"
Example	The ≤ operator acting on the set of integers is a partial order, denoted by (**Z**, ≤). The relation is: 1. Reflexive (x ≤ x) 2. Anti-symmetric (if x ≤ y and y ≤ x then x = y). 3. Transitive (x ≤ y and y ≤ z implies that x ≤ z).	 **x ≤ y**
Poset	Partially Ordered Set The domain along with a partial order defined on it is denoted (A, ⪯).	(A, ⪯)
Comparable	If x ⪯ y or y ⪯ x.	Example: (**Z**, ≤)
Incomparable	Not comparable. Neither x ⪯ y nor y ⪯ x. Either they are <u>not connected</u> at all by a path of line segments or the only paths between x and y require a <u>change in direction</u> from up to down or from down to up.	c and f are incomparable. 
Total Order	If every two elements in the domain are comparable.	Example: (**Z**, ≤)
Minimal	An element x is a **minimal** element if there is no y ≠ x such that y ⪯ x. Vertex x has in-degree = 0.	All edges are leaving the vertex. 
Maximal	An element x is a **maximal** element if there is no y ≠ x such that x ⪯ y. Vertex x has out-degree = 0.	All edges are entering the vertex. 
Hasse Diagram	Ordered from top to bottom to identify if comparable. Incomparable if up then down or vice versa is needed. Incomparable if "air gaped".	

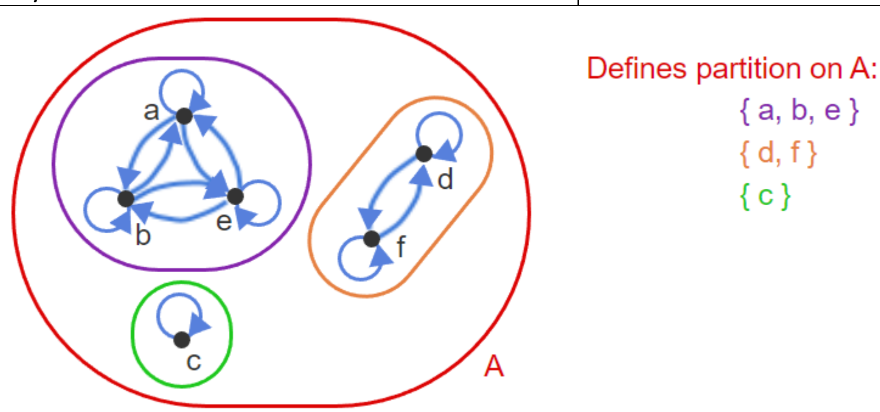
Strict Orders

Term	Description	Additional
Strict Order	A relation R on a set A that is **transitive** and **anti-reflexive**. Every strict order is anti-symmetric (assumed). A strict order acts like a < operator on the elements of A.	$aRb = a \prec b$ "a is less than b"
Example	The real numbers (**R**) along with the < relation is a strict order. The relation is: 1. Transitive (if a < b and b < c, then a < c) 2. Anti-reflexive (there is no real a such that a < a)	Examples: (**R**, <) (P(A), ⊂)
Comparable **Incomparable** **Total Order** **Minimal** **Maximal**	Same as a partial order above, except: Partial Order: $\preceq$ Strict Order: $\prec$	The arrow diagram for a strict order is basically an arrow diagram for a partial order without the self-loops.

Directed Acyclic Graphs (DAG)

Term	Description	Additional
DAG	Directed Acyclic Graph (DAG) A directed graph (digraph) that has no directed cycles or positive length cycles. Note that since a single vertex is a cycle of zero length. Acyclic = No Cycles	
Example	Useful for representing precedence relationships or constraints.	College course prerequisites graph or software module dependencies.
Theorem: DAGs and Strict Orders	Let G be a directed graph. G has no positive length cycles if and only if G^+ is a strict order.	If G is a DAG, then G^+ is a strict order. If G^+ is a strict order, then G is a DAG.
Topological Sort	If there is an edge (u, v), then u appears earlier than v.	A topological sort for a DAG G is also a topological sort for G^+.
Example	One way to construct a topological sort for a DAG G is to: 1) Pick a vertex x with in-degree 0 and remove x from G. 2) Then pick another vertex with in-degree 0 from among the remaining vertices. 3) Keep selecting vertices until there are no vertices left.	

Equivalence Relations

Term	Description	Additional
Equivalence Relation	A relation R is an equivalence relation if R is **reflexive**, **symmetric**, and **transitive**.	aRb = a ~ b "a is equivalent to b"
Example	The domain is the set of all people. Define relation B such that xBy if person x and person y have the same birthday. The relation B is: 1. Reflexive since every person has the same birthday as himself/herself. 2. Symmetric because if x has the same birthday as y, then y has the same birthday as x. 3. Transitive because if x and y share a birthday and y and z share a birthday, then x and z must also share a birthday.	
Equivalence Class	If A is the domain of an equivalence relation and a ∈ A, then [a] is defined to be the set of all x ∈ A such that a ~ x.	The set [a]. a ∈ A ⟶ [a] ⊆ A
Theorem: Structure of Equivalence Relations	Consider an equivalence relation on a set A. Let x, y ∈ A: • If x ~ y then [x] = [y] (identical) • If it is not the case that x ~ y, then [x] ∩ [y] = ∅ (completely disjoint)	The vertices of the network can be partitioned into sets of vertices that can all communicate with each other.
Partition	Consider an equivalence relation over a set A. The set of all distinct equivalence classes defines a **partition** of A. The term "distinct" means that if there are two equal equivalence classes [a] = [b], the set [a] is only included once.	Equivalence Relation ⟶ Equivalent Class ⟶ Partition ⟶ Set A
Example		
Pairwise Disjoint	The intersection of any pair of the sets is empty.	

Strong Connectivity	Strong connectivity is an equivalence relation on the set of vertices: 1. **Reflexive**: Every vertex v is strongly connected to itself. 2. **Symmetric**: If v is strongly connected to w, then w is strongly connected to v. 3. **Transitive**: If v is strongly connected to w and w is strongly connected to x, then v is also strongly connected to x.	

Sources:

- SNHU MAT 230 - Discrete Mathematics, zyBooks.
- See also "Harold's Undirected Graphs and Trees Cheat Sheet".
- See also pages 9 & 10 of "Harold's Sets Cheat Sheet" for Relations.

