Harold's AP Calculus Notes
Cheat Sheet
4 February 2024

Limits	
Definition of Limit Let f be a function defined on an open interval containing c and let L be a real number. The statement: $$\lim_{x \to c} f(x) = L$$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|x - c\| < \delta$, then $\|f(x) - L\| < \epsilon$ **Tip:** Direct substitution: Plug in $f(c)$ and see if it provides a legal answer. If so then $L = f(c)$.	
The Existence of a Limit The limit of $f(x)$ as x approaches c is L if and only if:	$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L$$
Definition of Continuity A function **f is continuous** at c if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x - c\| < \delta$ and $\|f(x) - f(c)\| < \varepsilon$. **Tip:** Rearrange $\|f(x) - f(c)\|$ to have $\|(x - c)\|$ as a factor. Since $\|x - c\| < \delta$ we can find an equation that relates both δ and ε together.	**Prove** that $f(x) = x^2 - 1$ is a continuous function. $$\begin{aligned}&\|f(x) - f(c)\|\\&= \|(x^2 - 1) - (c^2 - 1)\|\\&= \|x^2 - 1 - c^2 + 1\|\\&= \|x^2 - c^2\|\\&= \|(x + c)(x - c)\|\\&= \|(x + c)\| \; \|(x - c)\|\\&= \|x + c\| \; \|x - c\|\end{aligned}$$ Since $\|x + c\| \le \|2c\|$ $$\|f(x) - f(c)\| \le \|2c\| \; \|x - c\| < \varepsilon$$ So, given $\varepsilon > 0$, we can **choose $\delta = \left\|\dfrac{1}{2c}\right\| \varepsilon > 0$** in the Definition of Continuity. So, substituting the chosen δ for $\|x - c\|$ we get: $$\|f(x) - f(c)\| \le \|2c\| \left(\left\|\frac{1}{2c}\right\| \varepsilon \right) = \varepsilon$$ Since both conditions are met, $f(x)$ is continuous.
Two Special Trig Limits	$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Derivatives	(See Larson's 1-pager of common derivatives)
Definition of a Derivative of a Function (Slope Function)	$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ $$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$
Derivatives Notation	$\dfrac{dy}{dx}, y', f'(x), f^{(n)}(x), \dfrac{d}{dx}[f(x)], D_x[y]$
1. Chain Rule	$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$ $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$
2. Constant Rule	$$\frac{d}{dx}[c] = 0$$
3. Constant Multiple Rule	$$\frac{d}{dx}[cf(x)] = cf'(x)$$
4. Sum and Difference Rule	$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$
5. Product Rule	$$\frac{d}{dx}[fg] = fg' + gf'$$
6. Quotient Rule	$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$$
7. Power Rule	$$\frac{d}{dx}[x^n] = nx^{n-1}$$
8. General Power Rule	$$\frac{d}{dx}[u^n] = nu^{n-1}\,u' \text{ where } u = u(x)$$
9. Power Rule for x	$$\frac{d}{dx}[x] = 1 \ \ (\text{think } x = x^1 \text{ and } x^0 = 1)$$
10. Absolute Value	$$\frac{d}{dx}[|x|] = \frac{x}{|x|}$$
11. Natural Exponential Rule	$$\frac{d}{dx}[e^x] = e^x$$
12. General Natural Exponential Rule	$$\frac{d}{dx}[e^{g(x)}] = e^{g(x)} \cdot g'(x)$$
13. Exponential Rule	$$\frac{d}{dx}[a^x] = (\ln a) \cdot a^x$$
14. General Exponential Rule	$$\frac{d}{dx}[a^{g(x)}] = (\ln a) \cdot a^{g(x)} \cdot g'(x)$$
15. Natural Logarithm Rule	$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$
16. General Natural Logarithm Rule	$$\frac{d}{dx}[\ln f(x)] = \frac{1}{f(x)} \cdot f'(x)$$
17. Logorithm Rule	$$\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)\,x}$$
18. General Logorithm Rule	$$\frac{d}{dx}[\log_a f(x)] = \frac{1}{\ln x} \cdot \frac{f'(x)}{f(x)}$$

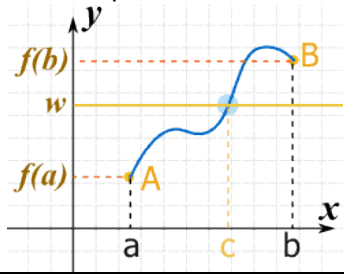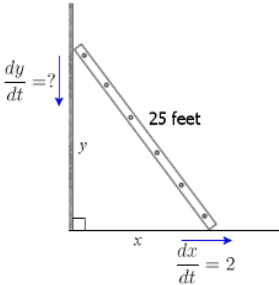|	
19. Sine	$$\frac{d}{dx}[sin(x)] = \cos(x)$$
20. Cosine	$$\frac{d}{dx}[cos(x)] = -sin(x)$$
21. Tangent	$$\frac{d}{dx}[tan(x)] = sec^2(x)$$
22. Cotangent	$$\frac{d}{dx}[cot(x)] = -csc^2(x)$$
23. Secant	$$\frac{d}{dx}[sec(x)] = sec(x)\,tan(x)$$
24. Cosecant	$$\frac{d}{dx}[csc(x)] = -csc(x)\,cot(x)$$
25. Arcsine	$$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$$
26. Arccosine	$$\frac{d}{dx}[\cos^{-1}(x)] = \frac{-1}{\sqrt{1-x^2}}$$
27. Arctangent	$$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$$
28. Arccotangent	$$\frac{d}{dx}[\cot^{-1}(x)] = \frac{-1}{1+x^2}$$
29. Arcsecant	$$\frac{d}{dx}[\sec^{-1}(x)] = \frac{1}{|x|\sqrt{x^2-1}}$$
30. Arccosecant	$$\frac{d}{dx}[\csc^{-1}(x)] = \frac{-1}{|x|\sqrt{x^2-1}}$$
31. Hyperbolic Sine $\left(\frac{e^x-e^{-x}}{2}\right)$	$$\frac{d}{dx}[sinh(x)] = \cosh(x)$$
32. Hyperbolic Cosine $\left(\frac{e^x+e^{-x}}{2}\right)$	$$\frac{d}{dx}[cosh(x)] = sinh(x)$$
33. Hyperbolic Tangent	$$\frac{d}{dx}[tanh(x)] = sech^2(x)$$
34. Hyperbolic Cotangent	$$\frac{d}{dx}[coth(x)] = -csch^2(x)$$
35. Hyperbolic Secant	$$\frac{d}{dx}[sech(x)] = -sech(x)\,tanh(x)$$
36. Hyperbolic Cosecant	$$\frac{d}{dx}[csch(x)] = -csch(x)\,coth(x)$$
37. Hyperbolic Arcsine	$$\frac{d}{dx}[\sinh^{-1}(x)] = \frac{1}{\sqrt{x^2+1}}$$
38. Hyperbolic Arccosine	$$\frac{d}{dx}[\cosh^{-1}(x)] = \frac{1}{\sqrt{x^2-1}}$$
39. Hyperbolic Arctangent	$$\frac{d}{dx}[\tanh^{-1}(x)] = \frac{1}{1-x^2}$$
40. Hyperbolic Arccotangent	$$\frac{d}{dx}[\coth^{-1}(x)] = \frac{1}{1-x^2}$$
41. Hyperbolic Arcsecant	$$\frac{d}{dx}[\text{sech}^{-1}(x)] = \frac{-1}{x\sqrt{1-x^2}}$$

42. Hyperbolic Arccosecant	$\dfrac{d}{dx}\left[\operatorname{csch}^{-1}(x)\right] = \dfrac{-1}{|x|\sqrt{1+x^2}}$
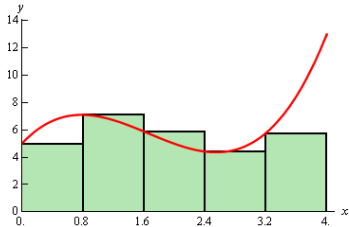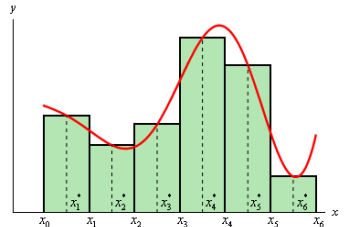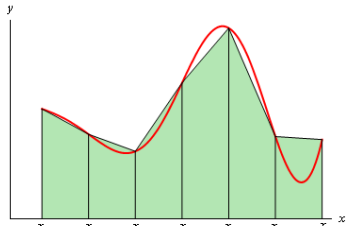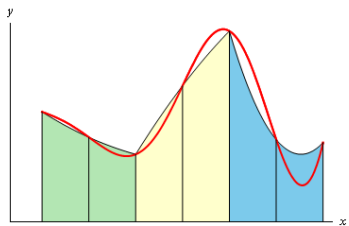
Physics	Translational Motion
Position Function	$s(t) = \dfrac{1}{2}\boldsymbol{g}t^2 + \boldsymbol{v}_0 t + \boldsymbol{s}_0$
Velocity Function	$v(t) = s'(t) = \boldsymbol{g}t + \boldsymbol{v}_0$
Acceleration Function	$a(t) = v'(t) = s''(t)$
Jerk Function	$j(t) = a'(t) = v''(t) = s^{(3)}(t)$
Gravity (g)	$\boldsymbol{g} \approx -9.81\dfrac{m}{s^2}\ or -32.2\dfrac{ft}{s^2}$

Analyzing the Graph of a Function	(See Harold's Illegals and Graphing Rationals Cheat Sheet)
x-Intercepts (Zeros or Roots)	$f(x) = 0$
y-Intercept	$f(0) = y$
Domain	Valid x values
Range	Valid y values
Continuity	No division by 0, no negative square roots or logs
Vertical Asymptotes (VA)	x = division by 0 or undefined
Horizontal Asymptotes (HA)	$\lim\limits_{x \to \infty^-} f(x) \to y$ and $\lim\limits_{x \to \infty^+} f(x) \to y$
Infinite Limits at Infinity	$\lim\limits_{x \to \infty^-} f(x) \to \infty$ and $\lim\limits_{x \to \infty^+} f(x) \to \infty$
Differentiability	Limit from both directions arrives at the same slope
Relative Extrema	Create a table with domains: $f(x), f'(x), f''(x)$
Concavity	If $f''(x) \to +$, then cup up $\cup$ If $f''(x) \to -$, then cup down $\cap$
Points of Inflection	$f''(x) = 0$ (concavity changes)

Graphing with Derivatives	
Test for Increasing and Decreasing Functions	1. If $f'(x) > 0$, then f is increasing (slope up) ↗ 2. If $f'(x) < 0$, then f is decreasing (slope down) ↘ 3. If $f'(x) = 0$, then f is constant (zero slope) →
First Derivative Test	1. If $f'(x)$ changes from $-$ to $+$ at c, then f has a *relative minimum* at *(c, f(c))* 2. If $f'(x)$ changes from $+$ to $-$ at c, then f has a *relative maximum* at *(c, f(c))* 3. If $f'(x)$, is $+$ c $+$ or $-$ c $-$, then $f(c)$ is neither
Second Deriviative Test Let $f'(c)=0$, and $f''(x)$ exists, then	1. If $f''(x) > 0$, then f has a relative minimum at $(c, f(c))$ 2. If $f''(x) < 0$, then f has a relative maximum at $(c, f(c))$ 3. If $f''(x) = 0$, then the test fails (See 1st derivative test)
Test for Concavity	1. If $f''(x) > 0$ for all x, then the graph is concave up $\cup$ 2. If $f''(x) < 0$ for all x, then the graph is concave down $\cap$
Points of Inflection Change in concavity	If $(c, f(c))$ is a point of inflection of $f(x)$, then either 1. $f''(c) = 0$ or 2. $f''(x)$ does not exist at $x = c$

Tangent Lines	
Genreal Form	$ax + by + c = 0$
Slope-Intercept Form	$y = mx + b$
Point-Slope Form	$y - y_0 = m(x - x_0)$ $where\ m = f'(x_0)\ at\ point\ (x_0, y_0)$
Calculus Form	$f(x) = f'(c)(x - c) + f(c)$
Slope	$m = \dfrac{rise}{run} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x} \to \dfrac{dy}{dx} = f'(x)$

Differentiation & Differentials	
Rolle's Theorem f is <u>continuous</u> on the closed interval [a,b], and f is <u>differentiable</u> on the open interval (a,b).	If $f(a) = f(b)$, then there exists at least one number c in *(a,b)* such that $f'(c) = 0$.
Mean Value Theorem If f meets the conditions of Rolle's Theorem, then you can find '*c*'.	$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\Delta y}{\Delta x}$$ $$f(b) = f(a) + (b - a)f'(c)$$
Intermediate Value Theorem f is a continuous function with an interval, [a, b], as its domain.	If f takes **values** $f(a)$ and $f(b)$ at each end of the interval, then it also takes any **value** between $f(a)$ and $f(b)$ at some point within the interval.
Calculating Differentials (Tanget line approximation)	$$f(x + \Delta x) \approx f(x) + \Delta y = f(x) + f'(x)\,\Delta x$$ $$dy = f'(x)\,dx \quad \text{so} \quad \Delta y = f'(x)\,\Delta x$$ $$\text{Relative Error} = \frac{\Delta f}{f} \text{ in \%}$$ Example: $\sqrt[4]{82} \to f(x) = \sqrt[4]{x}, f(x + \Delta x) = f(81 + 1)$
Newton's Method (Finds zeros of *f*, or finds *c* if *f(c) = 0*.)	$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ Example: $\sqrt[4]{82} \to f(x) = x^4 - 82 = 0, x_n = 3$
Related Rates 	Steps to solve: 1. Identify the known variables and rates of change. $$x = 15\,m; \; y = 20\,m; \; x' = 2\frac{m}{s}; \; y' = ?$$ 2. Construct an equation relating these quantities. $$x^2 + y^2 = r^2$$ 3. Differentiate both sides of the equation. $$2xx' + 2yy' = 0$$ 4. Solve for the desired rate of change. $$y' = -\frac{x}{y}\,x'$$ 5. Substitute the known rates of change and quantities into the equation. $$y' = -\frac{15}{20} \bullet 2 = \frac{3}{2}\frac{m}{s}$$
L'Hôpital's Rule	$$If \; \lim_{x \to c} f(x) = \lim_{x \to c} \frac{P(x)}{Q(x)} \text{ and}$$ $$\left\{\frac{0}{0}, \frac{\infty}{\infty}, 0 \bullet \infty, 1^\infty, 0^0, \infty^0, \infty - \infty\right\}, \text{but not } \{0^\infty\},$$ $$\text{then } \lim_{x \to c} \frac{P(x)}{Q(x)} = \lim_{x \to c} \frac{P'(x)}{Q'(x)} = \lim_{x \to c} \frac{P''(x)}{Q''(x)} = \cdots$$

Numerical Methods	
Riemann Sum	$$P_0(x) = \int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = max\{\Delta x_i\}$ Types: • Left Sum (LHS) • Middle Sum (MHS) • Right Sum (RHS)
Midpoint Rule (Middle Sum)	$$P_0(x) = \int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x =$$ $$\Delta x[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + \cdots + f(\bar{x}_n)]$$ where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = midpoint\ of\ [x_{i-1}, x_i]$ Error Bounds: $|E_M| \le \frac{K(b-a)^3}{24n^2}$
Trapezoidal Rule	$$P_1(x) = \int_a^b f(x)\,dx \approx$$ $$\frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)]$$ where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ Error Bounds: $|E_T| \le \frac{K(b-a)^3}{12n^2}$
Simpson's Rule	$$P_2(x) = \int_a^b f(x)dx \approx$$ $$\frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$ Where n is even and $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ Error Bounds: $|E_S| \le \frac{K(b-a)^5}{180n^4}$
TI-84 Plus	[MATH] fnInt(f(x),x,a,b), [MATH] [1] [ENTER] Example: [MATH] fnInt(x^2,x,0,1) $$\int_0^1 x^2\,dx = \frac{1}{3}$$
TI-Nspire CAS	[MENU] [4] Calculus [3] Integral [TAB] [TAB] [X] [^] [2] [TAB] [TAB] [X] [ENTER] Shortcut: [ALPHA] [WINDOWS] [4]

Integration	(See Harold's Fundamental Theorem of Calculus Cheat Sheet)
Basic Integration Rules Integration is the "inverse" of differentiation, and vice versa.	$$\int f'(x)\, dx = f(x) + C$$ $$\frac{d}{dx}\int f(x)\, dx = f(x)$$
$f(x) = 0$	$$\int 0\, dx = C$$
$f(x) = k = kx^0$	$$\int k\, dx = kx + C$$
1. The Constant Multiple Rule	$$\int k\,f(x)\, dx = k\int f(x)\, dx$$
2. The Sum and Difference Rule	$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$
The Power Rule $f(x) = kx^n$	$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \text{where } n \neq -1$$ $$\text{If } n = -1, \text{then } \int x^{-1}\, dx = \ln|x| + C$$
The General Power Rule	$$\text{If } u = g(x), \text{and } u' = \frac{d}{dx}g(x) \text{ then}$$ $$\int u^n\, u'\, dx = \frac{u^{n+1}}{n+1} + C, where\ n \neq -1$$
Reimann Sum	$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad \text{where } x_{i-1} \leq c_i \leq x_i$$ $$\|\Delta\| = \Delta x = \frac{b-a}{n}$$
Definition of a Definite Integral Area under curve	$$\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\, dx$$
Swap Bounds	$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$
Additive Interval Property	$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$
The Fundamental Theorem of Calculus	$$\int_a^b f(x)\, dx = F(b) - F(a)$$
The Second Fundamental Theorem of Calculus	$$\frac{d}{dx}\int_a^x f(t)\, dt = f(x)$$ $$\frac{d}{dx}\int_a^{g(x)} f(t)\, dt = f\big(g(x)\big)g'(x)$$ $$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\, dt = f\big(h(x)\big)h'(x) - f\big(g(x)\big)g'(x)$$
Mean Value Theorem for Integrals	$$\int_a^b f(x)\, dx = f(c)(b-a) \quad \text{Find 'c'.}$$
The Average Value for a Function	$$\frac{1}{b-a}\int_a^b f(x)\, dx$$

Integration Methods	
1. Memorized	See Larson's 1-pager of common integrals
2. U-Substitution	$$\int f(g(x))g'(x)dx = F(g(x)) + C$$ Set $u = g(x)$, then $du = g'(x)\, dx$ $$\int f(u)\, du = F(u) + C$$ $u = \underline{\quad}$ $du = \underline{\quad} dx$
3. Integration by Parts	$$\int u\, dv = uv - \int v\, du$$ $u = \underline{\quad}$ $v = \underline{\quad}$ $du = \underline{\quad}$ $dv = \underline{\quad}$ Pick 'u' using the **LIATE** Rule: **L – Logarithmic** : $\ln x$, $\log_b x$ **I – Inverse Trig.**: $\tan^{-1} x$, $\sec^{-1} x$, $etc.$ **A – Algebraic**: x^2, $3x^{60}$, $etc.$ **T – Trigonometric**: $\sin x$, $\tan x$, $etc.$ **E – Exponential**: e^x, 19^x
4. Partial Fractions	$$\int \frac{P(x)}{Q(x)}\, dx$$ where $P(x)$ and $Q(x)$ are polynomials **Case 1:** If degree of $P(x) \geq Q(x)$ then do long division first **Case 2:** If degree of $P(x) < Q(x)$ then do partial fraction expansion
5a. Trig Substitution for $\sqrt{a^2 - x^2}$	$$\int \sqrt{a^2 - x^2}\, dx$$ Substutution: $x = a \sin \theta$ Identity: $1 - sin^2\,\theta = cos^2\,\theta$
5b. Trig Substitution for $\sqrt{x^2 - a^2}$	$$\int \sqrt{x^2 - a^2}\, dx$$ Substutution: $x = a \sec \theta$ Identity: $sec^2\,\theta - 1 = tan^2\,\theta$
5c. Trig Substitution for $\sqrt{x^2 + a^2}$	$$\int \sqrt{x^2 + a^2}\, dx$$ Substutution: $x = a \tan \theta$ Identity: $tan^2\,\theta + 1 = sec^2\,\theta$
6. Table of Integrals	CRC Standard Mathematical Tables book
7. Computer Algebra Systems (CAS)	TI-Nspire CX CAS Graphing Calculator TI –Nspire CAS iPad app
8. Numerical Methods	Riemann Sum, Midpoint Rule, Trapezoidal Rule, Simpson's Rule, TI-84, etc.
9. WolframAlpha	Google of mathematics. Shows steps. Free. www.wolframalpha.com

Partial Fractions	(See Harold's Partial Fractions Cheat Sheet)
Condition	$$f(x) = \frac{P(x)}{Q(x)}$$ where $P(x)$ and $Q(x)$ are polynomials and degree of $P(x) < Q(x)$ If degree of $P(x) \geq Q(x)$ then do long division first
Example Expansion	$$\frac{P(x)}{(ax+b)(cx+d)^2(ex^2+fx+g)}$$ $$= \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2} + \frac{Dx+E}{(ex^2+fx+g)}$$
Typical Solution	$$\int \frac{a}{x+b}\,dx = a\,\ln|x+b| + C$$

Sequences & Series	(See Harold's Series Cheat Sheet)
Sequence	$$\lim_{n\to\infty} a_n = L \quad \text{(Limit)}$$ Example: $(a_n, a_{n+1}, a_{n+2}, \dots)$
Geometric Series	$$S = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$$ only if $|r| < 1$ where r is the radius of convergence and $(-r, r)$ is the interval of convergence

Convergence Tests	(See Harold's Series Convergence Tests Cheat Sheet)	
Series Convergence Tests	1. Divergence or n^{th} Term	6. Ratio
	2. Geometric Series	7. Root
	3. p-Series	8. Direct Comparison
	4. Alternating Series	9. Limit Comparison
	5. Integral	10. Telescoping Series

Taylor Series	(See Harold's Taylor Series Cheat Sheet)
Taylor Series	$$f(x) = P_n(x) + R_n(x)$$ $$= \sum_{n=0}^{+\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(x^*)}{(n+1)!}(x-c)^{n+1}$$ where $x \leq x^* \leq c$ (worst case scenario x^*) and $\lim_{x\to+\infty} R_n(x) = 0$