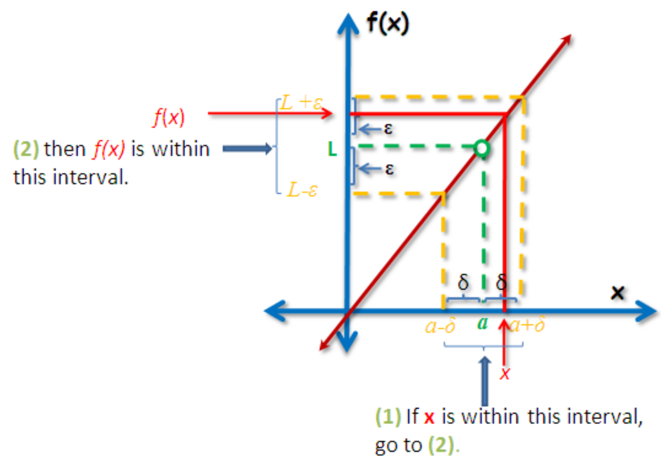
Harold's AP Calculus Notes
Cheat Sheet
13 December 2022

<table>
<tr><th>Limits</th><th></th></tr>
<tr><td>Definition of Limit
Let f be a function defined on an open interval containing a and let L be a real number. The statement:
$$\lim_{x \to a} f(x) = L$$
means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that
$$\text{if } 0 < |x - a| < \delta,$$
$$\text{then } |f(x) - L| < \epsilon$$
Tip :
Direct substitution: Plug in $f(a)$ and see if it provides a legal answer. If so then $L = f(a)$.</td>
<td>
</td></tr>
<tr><td>The Existence of a Limit
The limit of $f(x)$ as x approaches a is L if and only if:</td>
<td>$$\lim_{x \to a^-} f(x) = L$$ $$\lim_{x \to a^+} f(x) = L$$</td></tr>
<tr><td>Definition of Continuity

A function f is continuous at c if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - c| < \delta$ and $|f(x) - f(c)| < \varepsilon$.

Tip: Rearrange $|f(x) - f(c)|$ to have $|(x - c)|$ as a factor. Since $|x - c| < \delta$ we can find an equation that relates both δ and ε together.</td>
<td><u>Prove</u> that $f(x) = x^2 - 1$ is a continuous function.
$$\begin{aligned} &|f(x) - f(c)| \\ &= |(x^2 - 1) - (c^2 - 1)| \\ &= |x^2 - 1 - c^2 + 1| \\ &= |x^2 - c^2| \\ &= |(x + c)(x - c)| \\ &= |(x + c)|\ |(x - c)| \end{aligned}$$
Since $|(x + c)| \leq |2c|$
$$|f(x) - f(c)| \leq |2c||(x - c)| < \varepsilon$$
So, given $\varepsilon > 0$, we can choose $\boldsymbol{\delta} = \left|\frac{1}{2c}\right| \boldsymbol{\varepsilon} > \mathbf{0}$ in the Definition of Continuity. So, substituting the chosen δ for $|(x - c)|$ we get:
$$|f(x) - f(c)| \leq |2c| \left(\left|\frac{1}{2c}\right| \varepsilon \right) = \varepsilon$$
Since both conditions are met, $f(x)$ is continuous.</td></tr>
<tr><td>Two Special Trig Limits</td>
<td>$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$ $$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$</td></tr>
</table>

Derivatives	(See Larson's 1-pager of common derivatives)
Definition of a Derivative of a Function (Slope Function)	$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$
Derivatives Notation	$\frac{dy}{dx}, y', f'(x),\ f^{(n)}(x), \frac{d}{dx}[f(x)],\ D_x[y]$
1. Chain Rule	$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$ $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$
2. Constant Rule	$\frac{d}{dx}[c] = 0$
3. Constant Multiple Rule	$\frac{d}{dx}[cf(x)] = cf'(x)$
4. Sum and Difference Rule	$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
5. Product Rule	$\frac{d}{dx}[fg] = fg' + g\,f'$
6. Quotient Rule	$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$
7. Power Rule	$\boldsymbol{\frac{d}{dx}[x^n] = nx^{n-1}}$
8. General Power Rule	$\frac{d}{dx}[u^n] = nu^{n-1}\,u'$ where $u = u(x)$
9. Power Rule for x	$\frac{d}{dx}[x] = 1$ (think $x = x^1$ and $x^0 = 1$)
10. Absolute Value	$\frac{d}{dx}[\lvert x \rvert] = \frac{x}{\lvert x \rvert}$
11. Natural Exponential Rule	$\frac{d}{dx}[e^x] = e^x$
12. General Natural Exponential Rule	$\frac{d}{dx}[e^{g(x)}] = e^{g(x)} \cdot g'(x)$
13. Exponential Rule	$\frac{d}{dx}[a^x] = (ln\,a) \cdot a^x$
14. General Exponential Rule	$\frac{d}{dx}[a^{g(x)}] = (ln\,a) \cdot a^{g(x)} \cdot g'(x)$
15. Natural Logorithm Rule	$\frac{d}{dx}[ln\,x] = \frac{1}{x}$
16. General Natural Logorithm Rule	$\frac{d}{dx}[ln\,f(x)] = \frac{1}{f(x)} \cdot f'(x)$
17. Logorithm Rule	$\frac{d}{dx}[log_a\,x] = \frac{1}{(ln\,a)\,x}$
18. General Logorithm Rule	$\frac{d}{dx}[log_a\,f(x)] = \frac{1}{ln\,x} \cdot \frac{f'(x)}{f(x)}$

19. Sine	$\frac{d}{dx}[sin(x)] = \cos(x)$
20. Cosine	$\frac{d}{dx}[cos(x)] = -sin(x)$
21. Tangent	$\frac{d}{dx}[tan(x)] = sec^2(x)$
22. Cotangent	$\frac{d}{dx}[cot(x)] = -csc^2(x)$
23. Secant	$\frac{d}{dx}[sec(x)] = sec(x)\,tan(x)$
24. Cosecant	$\frac{d}{dx}[csc(x)] = -\,csc(x)\,cot(x)$
25. Arcsine	$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$
26. Arccosine	$\frac{d}{dx}[\cos^{-1}(x)] = \frac{-1}{\sqrt{1-x^2}}$
27. Arctangent	$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$
28. Arccotangent	$\frac{d}{dx}[\cot^{-1}(x)] = \frac{-1}{1+x^2}$
29. Arcsecant	$\frac{d}{dx}[\sec^{-1}(x)] = \frac{1}{\lvert x\rvert\,\sqrt{x^2-1}}$
30. Arccosecant	$\frac{d}{dx}[\csc^{-1}(x)] = \frac{-1}{\lvert x\rvert\,\sqrt{x^2-1}}$
31. Hyperbolic Sine $\left(\frac{e^x-e^{-x}}{2}\right)$	$\frac{d}{dx}[sinh(x)] = \cosh(x)$
32. Hyperbolic Cosine $\left(\frac{e^x+e^{-x}}{2}\right)$	$\frac{d}{dx}[cosh(x)] = sinh(x)$
33. Hyperbolic Tangent	$\frac{d}{dx}[tanh(x)] = sech^2(x)$
34. Hyperbolic Cotangent	$\frac{d}{dx}[coth(x)] = -csch^2(x)$
35. Hyperbolic Secant	$\frac{d}{dx}[sech(x)] = -\,sech(x)\,tanh(x)$
36. Hyperbolic Cosecant	$\frac{d}{dx}[csch(x)] = -\,csch(x)\,coth(x)$
37. Hyperbolic Arcsine	$\frac{d}{dx}[\sinh^{-1}(x)] = \frac{1}{\sqrt{x^2+1}}$
38. Hyperbolic Arccosine	$\frac{d}{dx}[\cosh^{-1}(x)] = \frac{1}{\sqrt{x^2-1}}$
39. Hyperbolic Arctangent	$\frac{d}{dx}[\tanh^{-1}(x)] = \frac{1}{1-x^2}$
40. Hyperbolic Arccotangent	$\frac{d}{dx}[\coth^{-1}(x)] = \frac{1}{1-x^2}$
41. Hyperbolic Arcsecant	$\frac{d}{dx}[\mathrm{sech}^{-1}(x)] = \frac{-1}{x\,\sqrt{1-x^2}}$

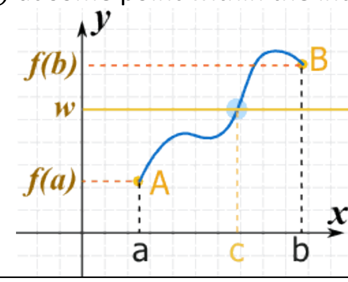
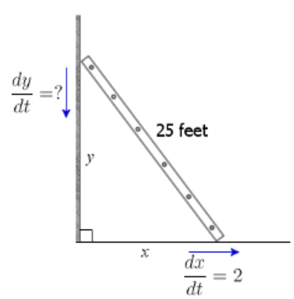
42. Hyperbolic Arccosecant	$\frac{d}{dx}[\operatorname{csch}^{-1}(x)] = \frac{-1}{\lvert x \rvert \sqrt{1+x^2}}$

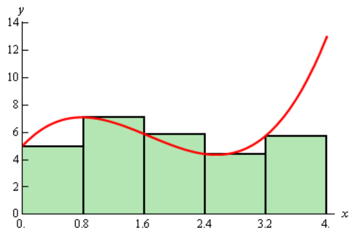
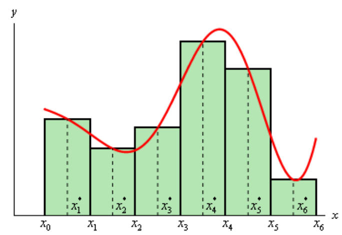
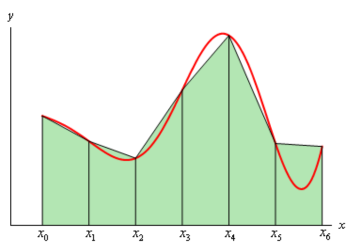
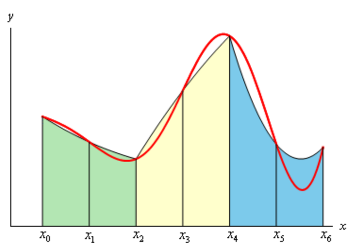
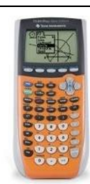
Physics	**Translational Motion**
Position Function	$s(t) = \frac{1}{2}gt^2 + v_0 t + s_0$
Velocity Function	$v(t) = s'(t) = gt + v_0$
Acceleration Function	$a(t) = v'(t) = s''(t)$
Jerk Function	$j(t) = a'(t) = v''(t) = s^{(3)}(t)$

Analyzing the Graph of a Function	(See Harold's Illegals and Graphing Rationals Cheat Sheet)
x-Intercepts (Zeros or Roots)	$f(x) = 0$
y-Intercept	$f(0) = y$
Domain	Valid x values
Range	Valid y values
Continuity	No division by 0, no negative square roots or logs
Vertical Asymptotes (VA)	x = division by 0 or undefined
Horizontal Asymptotes (HA)	$\lim_{x\to\infty^-} f(x) \to y$ and $\lim_{x\to\infty^+} f(x) \to y$
Infinite Limits at Infinity	$\lim_{x\to\infty^-} f(x) \to \infty$ and $\lim_{x\to\infty^+} f(x) \to \infty$
Differentiability	Limit from both directions arrives at the same slope
Relative Extrema	Create a table with domains: $f(x), f'(x), f''(x)$
Concavity	If $f''(x) \to +$, then cup up ⋃ If $f''(x) \to -$, then cup down ⋂
Points of Inflection	$f''(x) = 0$ (concavity changes)

Graphing with Derivatives	
Test for Increasing and Decreasing Functions	1. If $f'(x) > 0$, then f is increasing (slope up) ↗ 2. If $f'(x) < 0$, then f is decreasing (slope down) ↘ 3. If $f'(x) = 0$, then f is constant (zero slope) →
First Derivative Test	1. If $f'(x)$ changes from – to + at *c*, then f has a *relative minimum* at *(c, f(c))* 2. If $f'(x)$ changes from + to - at *c*, then f has a *relative maximum* at *(c, f(c))* 3. If $f'(x)$, is + *c* + or - *c* -, then $f(c)$ is neither
Second Deriviative Test Let *f ′(c)*=0, and *f ″(x)* exists, then	1. If $f''(x) > 0$, then f has a relative minimum at $(c, f(c))$ 2. If $f''(x) < 0$, then f has a relative maximum at $(c, f(c))$ 3. If $f''(x) = 0$, then the test fails (See 1^{st} derivative test)
Test for Concavity	1. If $f''(x) > 0$ for all x, then the graph is concave up ⋃ 2. If $f''(x) < 0$for all x, then the graph is concave down ⋂
Points of Inflection Change in concavity	If $(c, f(c))$ is a point of inflection of $f(x)$, then either 1. $f''(c) = 0$ or 2. $f''(x)$ does not exist at *x* = *c*

Tangent Lines	
Genreal Form	$ax + by + c = 0$
Slope-Intercept Form	$y = mx + b$
Point-Slope Form	$y - y_0 = m(x - x_0)$ $where\ m = f'(x_0)\ at\ point\ (x_0, y_0)$
Calculus Form	$y = f'(c)(x - c) + f(c)$
Slope	$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} \to \frac{dy}{dx} = f'(x)$

Differentiation & Differentials	
Rolle's Theorem *f* is <u>continuous</u> on the closed interval [a,b], and *f* is <u>differentiable</u> on the open interval (a,b).	If $f(a) = f(b)$, then there exists at least one number *c* in *(a,b)* such that $f'(c) = 0$.
Mean Value Theorem If *f* meets the conditions of Rolle's Theorem, then you can find '*c*'.	$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\Delta y}{\Delta x}$ $f(b) = f(a) + (b - a)f'(c)$
Intermediate Value Theorem f is a continuous function with an interval, [a, b], as its domain.	If f takes **values** $f(a)$ and $f(b)$ at each end of the interval, then it also takes any **value** between $f(a)$ and $f(b)$ at some point within the interval. 
Calculating Differentials (Tanget line approximation)	$f(x + \Delta x) \approx f(x) + \Delta y = f(x) + f'(x)\,\Delta x$ $dy = f'(x)\,dx \quad \text{so} \quad \Delta y = f'(x)\,\Delta x$ $\text{Relative Error} = \frac{\Delta f}{f} \text{ in } \%$ Example: $\sqrt[4]{82} \rightarrow f(x) = \sqrt[4]{x}, f(x + \Delta x) = f(81 + 1)$
Newton's Method (Finds zeros of *f*, or finds *c* if *f(c) = 0*.)	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ Example: $\sqrt[4]{82} \rightarrow f(x) = x^4 - 82 = 0, x_n = 3$
Related Rates 	Steps to solve: 1. Identify the known variables and rates of change. $x = 15\,m;\ y = 20\,m;\ x' = 2\frac{m}{s};\ y' = ?$ 2. Construct an equation relating these quantities. $x^2 + y^2 = r^2$ 3. Differentiate both sides of the equation. $2xx' + 2yy' = 0$ 4. Solve for the desired rate of change. $y' = -\frac{x}{y}x'$ 5. Substitute the known rates of change and quantities into the equation. $y' = -\frac{15}{20} \bullet 2 = \frac{3}{2}\frac{m}{s}$
L'Hôpital's Rule	$If\ \lim_{x \to c} f(x) = \lim_{x \to c} \frac{P(x)}{Q(x)}$ and $\left\{\frac{0}{0}, \frac{\infty}{\infty}, 0 \bullet \infty, 1^{\infty}, 0^0, \infty^0, \infty - \infty\right\}$, but not $\{0^{\infty}\}$, then $\lim_{x \to c} \frac{P(x)}{Q(x)} = \lim_{x \to c} \frac{P'(x)}{Q'(x)} = \lim_{x \to c} \frac{P''(x)}{Q''(x)} = \cdots$

Numerical Methods		
Riemann Sum		$P_0(x) = \int_a^b f(x)\,dx = \lim_{\Vert P \Vert \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$ where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and $\Delta x_i = x_i - x_{i-1}$ and $\Vert P \Vert = max\{\Delta x_i\}$ Types: • Left Sum (LHS) • Middle Sum (MHS) • Right Sum (RHS)
Midpoint Rule (Middle Sum)		$P_0(x) = \int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x =$ $\Delta x[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + \cdots + f(\bar{x}_n)]$ where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = midpoint\ of\ [x_{i-1}, x_i]$ Error Bounds: $\lvert E_M \rvert \le \frac{K(b-a)^3}{24n^2}$
Trapezoidal Rule		$P_1(x) = \int_a^b f(x)\,dx \approx$ $\frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_3) + \cdots + 2f(x_{n-1})$ $+ f(x_n)]$ where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ Error Bounds: $\lvert E_T \rvert \le \frac{K(b-a)^3}{12n^2}$
Simpson's Rule		$P_2(x) = \int_a^b f(x)dx \approx$ $\frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots$ $+ 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$ Where n is even and $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ Error Bounds: $\lvert E_S \rvert \le \frac{K(b-a)^5}{180n^4}$
TI-84 Plus		[MATH] fnInt(f(x),x,a,b), [MATH] [1] [ENTER] Example: [MATH] fnInt(x^2,x,0,1) $\int_0^1 x^2\,dx = \frac{1}{3}$
TI-Nspire CAS		[MENU] [4] Calculus [3] Integral [TAB] [TAB] [X] [^] [2] [TAB] [TAB] [X] [ENTER] Shortcut: [ALPHA] [WINDOWS] [4]

Integration	(See Harold's Fundamental Theorem of Calculus Cheat Sheet)
Basic Integration Rules Integration is the "inverse" of differentiation, and vice versa.	$\int f'(x)\,dx = f(x) + C$ $\frac{d}{dx}\int f(x)\,dx = f(x)$
$f(x) = 0$	$\int 0\,dx = C$
$f(x) = k = kx^0$	$\int k\,dx = kx + C$
1. The Constant Multiple Rule	$\int k\,f(x)\,dx = k\int f(x)\,dx$
2. The Sum and Difference Rule	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
The Power Rule $f(x) = kx^n$	$\boldsymbol{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C}$, where $n \neq -1$ $If\ n = -1$, then $\int x^{-1}\,dx = \ln\lvert x\rvert + C$
The General Power Rule	If $u = g(x)$, and $u' = \frac{d}{dx}g(x)$ then $\int u^n\,u'dx = \frac{u^{n+1}}{n+1} + C, where\ n \neq -1$
Reimann Sum	$\sum_{i=1}^{n} f(c_i)\,\Delta x_i,$ where $x_{i-1} \leq c_i \leq x_i$ $\lVert\Delta\rVert = \Delta x = \frac{b-a}{n}$
Definition of a Definite Integral Area under curve	$\lim_{\lVert\Delta\rVert\to 0}\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx$
Swap Bounds	$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
Additive Interval Property	$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
The Fundamental Theorem of Calculus	$\int_a^b f(x)\,dx = F(b) - F(a)$
The Second Fundamental Theorem of Calculus	$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f\big(g(x)\big)g'(x)$ $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f\big(h(x)\big)h'(x) - f\big(g(x)\big)g'(x)$
Mean Value Theorem for Integrals	$\int_a^b f(x)\,dx = f(c)(b-a)$ Find 'c'.
The Average Value for a Function	$\frac{1}{b-a}\int_a^b f(x)\,dx$

Integration Methods	
1. Memorized	See Larson's 1-pager of common integrals
2. U-Substitution	$\int f(g(x))g'(x)dx = F(g(x)) + C$ Set $u = g(x)$, then $du = g'(x)\,dx$ $\int f(u)\,du = F(u) + C$ $u = \_\_\_\_\_ \quad du = \_\_\_\_\_\,dx$
3. Integration by Parts	$\int u\,dv = uv - \int v\,du$ $u = \_\_\_\_\_ \quad v = \_\_\_\_\_$ $du = \_\_\_\_\_ \quad dv = \_\_\_\_\_$ Pick 'u' using the **LIATE** Rule: **L – Logarithmic** : $\ln x, \log_b x$ **I – Inverse Trig.**: $\tan^{-1} x, \sec^{-1} x, etc.$ **A – Algebraic**: $x^2, 3x^{60}, etc.$ **T – Trigonometric**: $\sin x, \tan x, etc.$ **E – Exponential**: $e^x, 19^x$
4. Partial Fractions	$\int \frac{P(x)}{Q(x)}\,dx$ where $P(x)$ and $Q(x)$ are polynomials **Case 1:** If degree of $P(x) \geq Q(x)$ then do long division first **Case 2:** If degree of $P(x) < Q(x)$ then do partial fraction expansion
5a. Trig Substitution for $\sqrt{a^2 - x^2}$	$\int \sqrt{a^2 - x^2}\,dx$ Substutution: $x = a\sin\theta$ Identity: $1 - sin^2\,\theta = cos^2\,\theta$
5b. Trig Substitution for $\sqrt{x^2 - a^2}$	$\int \sqrt{x^2 - a^2}\,dx$ Substutution: $x = a\sec\theta$ Identity: $sec^2\,\theta - 1 = tan^2\,\theta$
5c. Trig Substitution for $\sqrt{x^2 + a^2}$	$\int \sqrt{x^2 + a^2}\,dx$ Substutution: $x = a\tan\theta$ Identity: $tan^2\,\theta + 1 = sec^2\,\theta$
6. Table of Integrals	CRC Standard Mathematical Tables book
7. Computer Algebra Systems (CAS)	TI-Nspire CX CAS Graphing Calculator TI –Nspire CAS iPad app
8. Numerical Methods	Riemann Sum, Midpoint Rule, Trapezoidal Rule, Simpson's Rule, TI-84, etc.
9. WolframAlpha	Google of mathematics. Shows steps. Free. www.wolframalpha.com

Partial Fractions	(See Harold's Partial Fractions Cheat Sheet)
Condition	$f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials and degree of $P(x) < Q(x)$ If degree of $P(x) \geq Q(x)$ then do long division first
Example Expansion	$\frac{P(x)}{(ax+b)(cx+d)^2(ex^2+fx+g)}$ $= \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2} + \frac{Dx+E}{(ex^2+fx+g)}$
Typical Solution	$\int \frac{a}{x+b}\, dx = a\, ln\lvert x+b\rvert + C$

Sequences & Series	(See Harold's Series Cheat Sheet)
Sequence	$\lim_{n\to\infty} a_n = L$ (Limit) Example: $(a_n, a_{n+1}, a_{n+2}, \ldots)$
Geometric Series	$S = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$ only if $\lvert r\rvert < 1$ where r is the radius of convergence and $(-r, r)$ is the interval of convergence

Convergence Tests	(See Harold's Series Convergence Tests Cheat Sheet)	
Series Convergence Tests	1. Divergence or n^{th} Term 2. Geometric Series 3. p-Series 4. Alternating Series 5. Integral	6. Ratio 7. Root 8. Direct Comparison 9. Limit Comparison 10. Telescoping Series

Taylor Series	(See Harold's Taylor Series Cheat Sheet)
Taylor Series	$f(x) = P_n(x) + R_n(x)$ $= \sum_{n=0}^{+\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(x^*)}{(n+1)!}(x-c)^{n+1}$ where $x \leq x^* \leq c$ (worst case scenario x^*) and $\lim_{x\to+\infty} R_n(x) = 0$