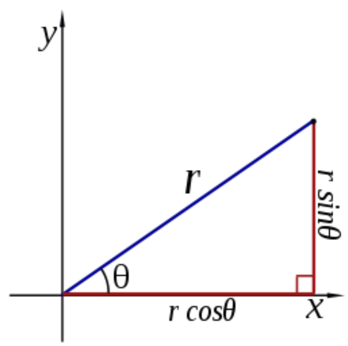
Harold's AP Calculus BC
Rectangular – Polar – Parametric
"Cheat Sheet"

28 August 2020

	Rectangular	Polar	Parametric
Point	$f(x) = y$ (x, y) (a, b)	(r, θ) or $r \angle \theta$ *Polar → Rect.* $x = r\cos\theta$ $y = r\sin\theta$ $\tan\theta = \frac{y}{x}$ *Rect. → Polar* $r^2 = x^2 + y^2$ $r = \pm\sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$	*Point (a,b) in Rectangular:* $x(t) = a$ $y(t) = b$ $< a, b >$ $t = 3^{rd}$ *variable, usually time, with 1 degree of freedom (df)*
Line	*Slope-Intercept Form:* $y = mx + b$ *Point-Slope Form:* $y - y_0 = m(x - x_0)$ *General Form:* $Ax + By + C = 0$ *Calculus Form:* $f(x) = f'(a)\,x + f(0)$		$< x, y > = < x_0, y_0 > + t < a, b >$ $< x, y > = < x_0 + at, y_0 + bt >$ *where* $< a, b > = < x_2 - x_1, y_2 - y_1 >$ $x(t) = x_0 + ta$ $y(t) = y_0 + tb$ $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{b}{a}$
Plane	$n_x(x - x_0)$ $+ n_y(y - y_0)$ $+ n_z(z - z_0) = 0$	*Vector Form:* $\boldsymbol{n} \bullet (\boldsymbol{r} - \boldsymbol{r}_0) = 0$	$\boldsymbol{r} = \boldsymbol{r}_0 + s\boldsymbol{v} + t\boldsymbol{w}$ *where:* • $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *are given vectors defining the plane* • $\boldsymbol{r_0}$ *is the vector of a fixed point on the plane*

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<tr><th></th><th>Rectangular</th><th>Polar</th><th>Parametric</th></tr>
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<td>Conics</td>
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General Equation for All Conics:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

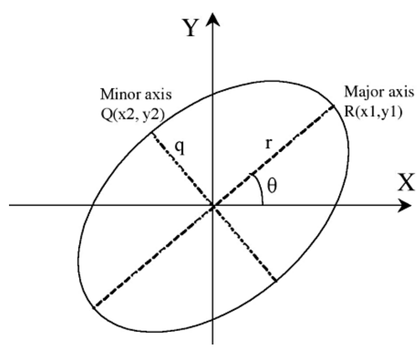
where

Line: $A = B = C = 0$
Circle: $A = C$ *and* $B = 0$
Ellipse: $AC > 0$
or $B^2 - 4AC < 0$
Parabola: $AC = 0$
or $B^2 - 4AC = 0$
Hyperbola: $AC < 0$
or $B^2 - 4AC > 0$
Note: If $A + C = 0$, *square hyperbola*

Rotation:
If B ≠ 0, then rotate coordinate system:

$$\cot 2\theta = \frac{A - C}{B}$$
$$x = x' \cos\theta - y' \sin\theta$$
$$y = y' \cos\theta + x' \sin\theta$$

New = (x', y'), Old = (x, y)
rotates through angle θ from x-axis



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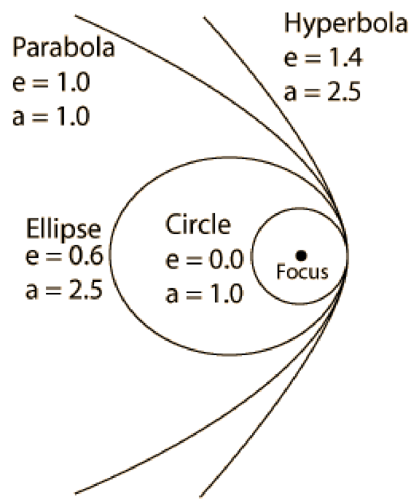
General Equation for All Conics:

$$r = \frac{p}{1 - e\cos\theta}$$

$$\text{where } p = \begin{cases} a\,(1 - e^2) \\ 2d \\ a\,(e^2 - 1) \end{cases} \text{for} \begin{cases} 0 \le e < 1 \\ e = 1 \\ e > 1 \end{cases}$$

p = semi-latus rectum
or the line segment running from the focus to the curve in a direction parallel to the directrix

Eccentricity:
Circle $e = 0$
Ellipse $0 < e < 1$
Parabola $e = 1$
Hyperbola $e > 1$



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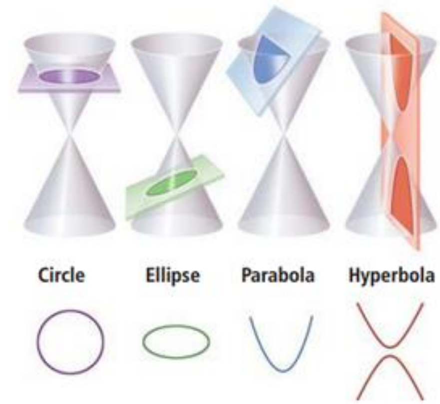

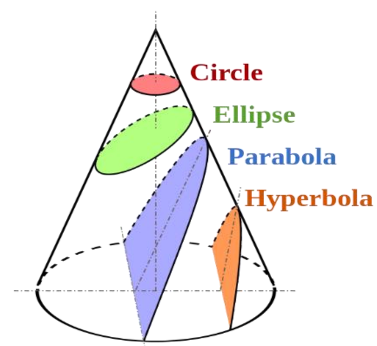

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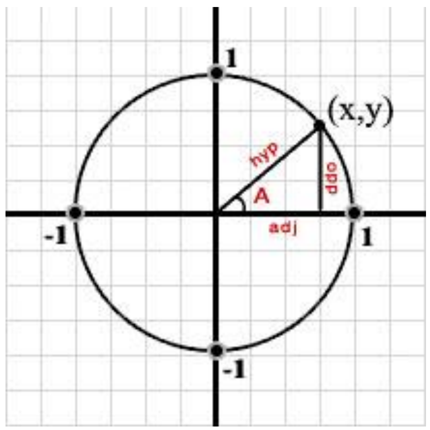
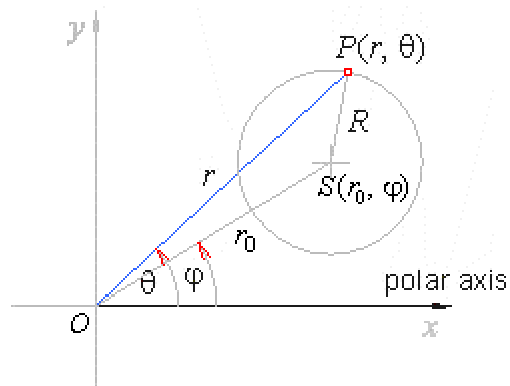
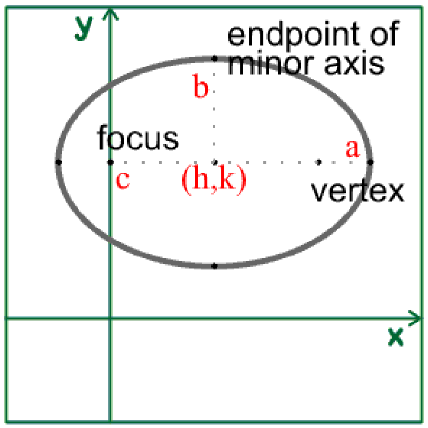
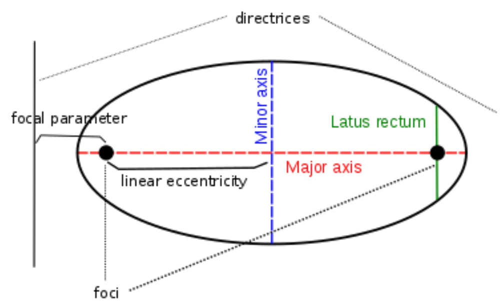
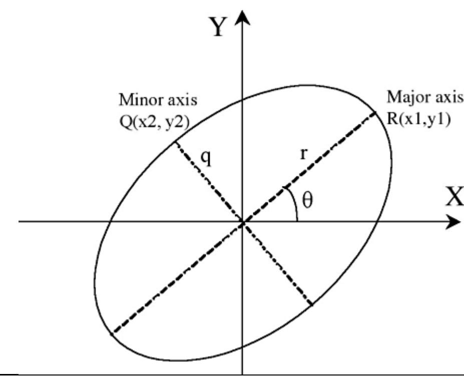
	Rectangular	Polar	Parametric
Circle	$x^2 + y^2 = r^2$ $(x-h)^2 + (y-k)^2 = r^2$ *Center:* (h, k) *Focus:* (h, k) 	*Centered at Origin:* *r = a (constant)* $\theta \;=\; \theta \quad [0, 2\pi]$ *or* $[0, 360°]$ *Centered at* (r_0, ϕ)*:* $r^2 + r_0^2 - 2rr_0\cos(\theta - \phi) = R^2$ *Hint: Law of Cosines* *or* $r = r_0\cos(\theta - \phi) + \sqrt{a^2 - r_0^2\, sin^2(\theta - \phi)}$ 	$x(t) = r\,cos(t) + h$ $y(t) = r\,sin(t) + k$ $[t_{min},\ t_{max}] = [0, 2\pi]$ *Center:* (h, k) *Focus:* (h, k)

	Rectangular	Polar	Parametric
Ellipse	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ *Center:* (h, k) *Vertices:* $(h \pm a, k)$ *and* $(h, k \pm b)$ *Foci:* $(h \pm c, k)$ *Focus length, c, from center:* $c = \sqrt{a^2 - b^2}$ 	*Ellipse:* $r = \frac{a\,(1-e^2)}{1+e\cos\theta}$ $for\ 0 < e < 1$ $where\ e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$ *relative to center (h, k)*  *Interesting Note:* *The <u>sum</u> of the distances from each focus to a point on the curve is constant.* $\lvert d_1 + d_2 \rvert = k$	$x(t) = a\cos(t) + h$ $y(t) = b\sin(t) + k$ $[t_{min}, t_{max}] = [0, 2\pi]$ *Center:* (h, k) *Rotated Ellipse:* $x(t) = a\cos t\cos\theta - b\sin t\sin\theta + h$ $y(t) = a\cos t\sin\theta + b\sin t\cos\theta + k$ θ *= the angle between the x-axis and the major axis of the ellipse* 

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<td>Parabola</td>
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Vertical Axis of Symmetry:

$x^2 = 4\,py$

$(x-h)^2 = 4p(y-k)$

Vertex: (h, k)

Focus: $(h, k+p)$

Directrix: $y = k - p$

Horizontal Axis of Symmetry:

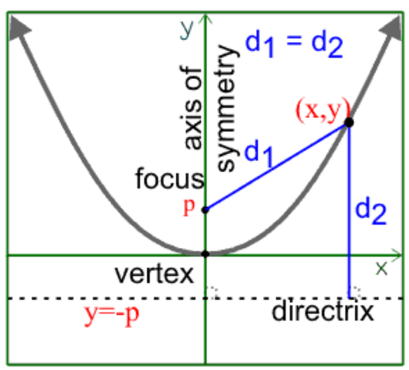
$y^2 = 4\,px$

$(y-k)^2 = 4p(x-h)$

Vertex: (h, k)

Focus: $(h+p, k)$

Directrix: $x = h - p$



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Vertical Axis of Symmetry:

$r = \dfrac{2d}{1 + e\cos\theta}$

Eccentricity: $e = 1$

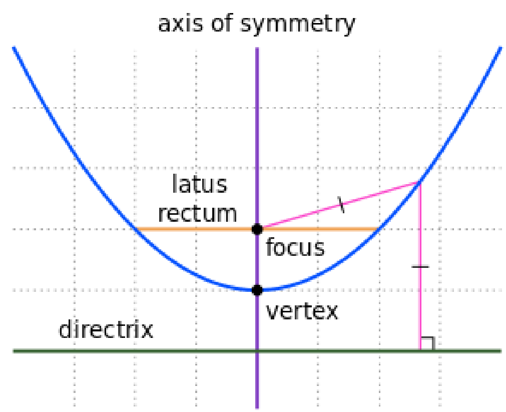
and $d = 2p$

Trigonometric Form:

$y = x^2$

$r\sin\theta = r^2\ \cos^2\theta$

$r = \dfrac{\sin\theta}{\cos^2\theta} = \tan\theta\sec\theta$



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Vertical Axis of Symmetry:

$x(t) = \ 2pt + \ h$

$y(t) = pt^2 + k$ *(opens upwards)*

$y(t) = -pt^2 - k$ *(opens downwards)*

$[t_{min},\ t_{max}] = [-c, c]$

Vertex: (h, k)

Horizontal Axis of Symmetry:

$y(t) = \ 2pt + \ k$

$x(t) = pt^2 + h$ *(opens to the right)*

$x(t) = -pt^2 - h$ *(opens to the left)*

$[t_{min},\ t_{max}] = [-c, c]$

Vertex: (h, k)

Projectile Motion:

$x(t) = x_0 + v_x t + \left(\frac{1}{2}\right) a_x t^2$

$y(t) = y_0 + v_{y0} t - 16t^2$ *feet*

$y(t) = y_0 + v_{y0} t - 4.9t^2$ *meters*

$v_x = v\cos\theta$

$v_y = v\sin\theta$

General Form:

$x = At^2 + \ Bt + C$

$y = Dt^2 + \ Et + F$

where A and D have the same sign

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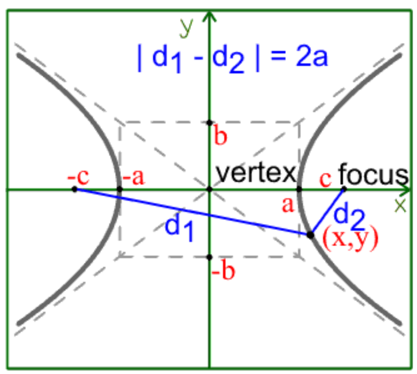
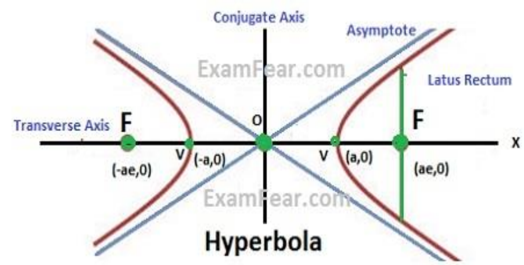
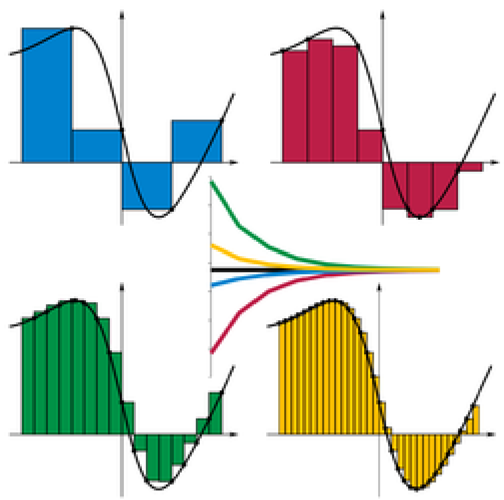
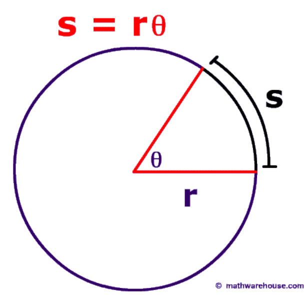
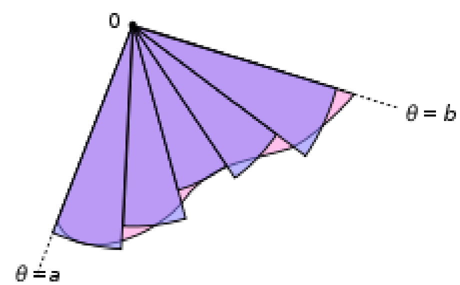
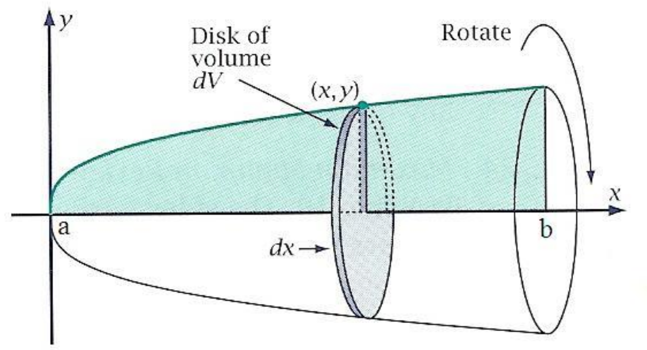
	Rectangular	Polar	Parametric
Hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ *Center:* (h, k) *Vertices:* $(h \pm a, k)$ *Foci:* $(h \pm c, k)$ *Focus length, c, from center:* $c = \sqrt{a^2 + b^2}$ 	*Vertical Axis of Symmetry:* $r = \frac{a\,(e^2 - 1)}{1 + e\cos\theta}\ \ for\ e > 1$ *Eccentricity:* $e > 1$ *where* $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} = \sec\theta > 1$ *relative to center (h, k)*  *p = semi-latus rectum* *or the line segment running from the focus to the curve in the directions* $\theta = \pm\frac{\pi}{2}$ *Interesting Note:* *The <u>difference</u> between the distances from each focus to a point on the curve is constant.* $\|d_1 - d_2\| = k$	*Left-Right Opening Hyperbola:* $x(t) = a\,sec(t) + h$ $y(t) = b\,tan(t) + k$ $[t_{min}, t_{max}] = [-c, c]$ *Vertex: (h, k)* *Up-Down Opening Hyperbola:* $x(t) = a\,tan(t) + h$ $y(t) = b\,sec(t) + k$ $[t_{min}, t_{max}] = [-c, c]$ *Vertex: (h, k)* *General Form:* $x(t) = At^2 + Bt + C$ $y(t) = Dt^2 + Et + F$ *where A and D have different signs*

	Rectangular	Polar	Parametric
1st Derivative	$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ $f'(x) = \frac{dy}{dx} = y' = D_x$	$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$ *Hint: Use Product Rule for* $y = r\sin\theta$ $x = r\cos\theta$	$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad provided\ \frac{dx}{dt} \neq 0$
2nd Derivative	$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y'' = D_{xx}$	$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$	$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}}$
Integral	*Fundamental Theorem of Calculus:* $F(x) = \int_a^b f(x)\,dx = F(b) - F(a)$		*Riemann Sum:* $S = \sum_{i-1}^{n} f(y_i)(x_i - x_{i-1})$ *Left Sum:* $S = \left(\frac{1}{n}\right)\left[f(a) + f\left(a + \frac{1}{n}\right) + f\left(a + \frac{2}{n}\right) + \cdots + f(b - \frac{1}{n})\right]$ *Middle Sum:* $S = \left(\frac{1}{n}\right)\left[f\left(a + \frac{1}{2n}\right) + f\left(a + \frac{3}{2n}\right) + \cdots + f(b - \frac{1}{2n})\right]$ *Right Sum:* $S = \left(\frac{1}{n}\right)\left[f\left(a + \frac{1}{n}\right) + f\left(a + \frac{2}{n}\right) + \cdots + f(b)\right]$

	Rectangular	Polar	Parametric
Inverse Functions	$f(f^{-1}(x)) = f^{-1}(f(x)) = x$ *Inverse Function Theorem:* $f^{-1}(f'(a)) = \frac{1}{f'(a)}$	$y = sin\,\theta$ $\theta = sin^{-1} y$ $y = cos\,\theta$ $\theta = cos^{-1} y$ $y = tan\,\theta$ $\theta = tan^{-1} y$ $y = csc\,\theta$ $\theta = csc^{-1} y$ $y = sec\,\theta$ $\theta = sec^{-1} y$ $y = cot\,\theta$ $\theta = cot^{-1} y$	$\theta = \arcsin y$ $\theta = \arccos\, y$ $\theta = arctan\, y$ $\theta = arccsc\, y$ $\theta = arcsec\, y$ $\theta = arccot\, y$
Arc Length	$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$ *Proof:* $\Delta s = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ $ds = \sqrt{dx^2 + dy^2}$ $ds = \sqrt{dx^2 + dy^2\left(\frac{dx^2}{dx^2}\right)}$ $ds = \sqrt{dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2}$ $ds = \sqrt{dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$ $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ $L = \int ds$	$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ *Circle:* $L = s = r\theta$ *Proof:* $L = (fraction\ of\ circumference) \cdot$ $\pi \cdot (diameter)$ $L = \left(\frac{\theta}{2\pi}\right)\pi\,(2r) = r\theta$  	$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$ *Proof:* $ds = \sqrt{dx^2 + dy^2}$ $ds = \sqrt{dx^2\left(\frac{dt^2}{dt^2}\right) + dy^2\left(\frac{dt^2}{dt^2}\right)}$ $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 dt^2}$ $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ $L = \int ds$
Perimeter	*Square:* $P = 4s$ *Rectangle:* $P = 2l + 2w$ *Triangle:* $P = a + b + c$ *Circle:* $C = \pi d = 2\pi r$ *Ellipse:* $C \approx \pi(a + b)$	*Ellipse:* $C \approx 2\pi\sqrt{\frac{a^2+b^2}{2}}$ $C \approx \pi\left[3(a + b) - \sqrt{(3a + b)(a + 3b)}\right]$ $C \approx \pi\,(a + b)\left(1 + \frac{3h}{10 + \sqrt{4 - 3h}}\right)$	*Ellipse:* $C = 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 sin^2\theta}\, d\theta$ $h = \frac{(a - b)^2}{(a + b)^2}$ & $k^2 = \left(1 - \frac{b^2}{a^2}\right)$

	Rectangular	Polar	Parametric
Area	*Square:* $A = s^2$ *Rectangle:* $A = lw$ *Rhombus:* $A = \frac{1}{2} ab$ *Parallelogram:* $A = Bh$ *Trapezoid:* $A = \frac{(B_1 + B_2)}{2} h$ *Kite:* $A = \frac{d_1 d_2}{2}$ *Triangle:* $A = \frac{1}{2} Bh = \frac{1}{2} ab \sin(C)$ *Heron's Formula:* $A = \sqrt{s(s-a)(s-b)(s-c)}$ $\text{where } s = \frac{a+b+c}{2}$ *Equilateral Triangle:* $A = \frac{1}{4}\sqrt{3}s^2$ *Frustum:* $A = \frac{1}{3}\left(\frac{B_1+B_2}{2}\right) h$ *Circle:* $A = \pi r^2$ *Circular Sector:* $A = \frac{1}{2} r^2\theta$ *Ellipse:* $A = \pi ab$	$A = \int_\alpha^\beta \frac{1}{2}[f(\theta)]^2 d\theta$ $\text{where } r = f(\theta)$ *Proof:* *Area of a sector:* $A = \int s\,dr = \int r\,\Delta\theta\,dr = \frac{1}{2} r^2\,\Delta\theta$ *where arc length* $s = r\,\Delta\theta$  	$A = \int_\alpha^\beta g(t)\, f'(t)\, dt$ $\text{where } f(t) = x \text{ and } g(t) = y$ *or* *x(t) = f(t) and y(t) = g(t)* *Simplified:* $A = \int_\alpha^\beta y(t)\, \frac{dx(t)}{dt}\, dt$ *Proof:* $\int_\alpha^b f(x)\, dx$ *y = f(x) = g(t)* $dx = \frac{df(t)}{dt} dt = f'(t)\, dt$
Lateral Surface Area	*Cylinder:* $SA = 2\pi rh$ *Cone:* $SA = \pi rl$ $S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx$	*For rotation about the x-axis:* $S = \int 2\pi y\, ds$ *For rotation about the y-axis:* $S = \int 2\pi x\, ds$ $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ $r = f(\theta), \quad \alpha \le \theta \le \beta$	*For rotation about the x-axis:* $S = \int 2\pi y\, ds$ *For rotation about the y-axis:* $S = \int 2\pi x\, ds$ $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ $\text{if } x = f(t), y = g(t), \alpha \le t \le \beta$

	Rectangular	Polar	Parametric
Total Surface Area	*Cube:* $SA = 6s^2$ *Rectangular Box:* $SA = 2lw + 2wh + 2hl$ *Regular Tetrahedron:* $SA = 2bh$ *Cylinder:* $SA = 2\pi r\,(r + h)$ *Cone:* $SA = \pi r^2 + \pi rl = \pi r\,(r + l)$ *Sphere:* $SA = 4\pi r^2$ *Ellipsoid:* $SA \approx 4\pi \left(\frac{a^p b^p + a^p c^p + b^p c^p}{3}\right)^{1/p}$ $Where\ p \approx 1.6075,\ \lvert Relative\ Error\rvert \leq 1.061\%$ *(Knud Thomsen's Formula)*		
Surface of Revolution	*For revolution about the x-axis:* $A = 2\pi \int_a^b f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$ *For revolution about the y-axis:* $A = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$	*For revolution about the x-axis:* $A = 2\pi r \int_\alpha^\beta \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$ *For revolution about the y-axis:* $A = 2\pi r \int_\alpha^\beta \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$	*For revolution about the x-axis:* $A = 2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$ *For revolution about the y-axis:* $A = 2\pi \int_a^b x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$
Volume	*Cube:* $V = s^3$ *Rectangular Prism:* $V = lwh$ *Cylinder:* $V = \pi r^2 h$ *Triangular Prism:* $V = Bh$ *Tetrahedron:* $V = \frac{1}{3} Bh$ *Pyramid:* $V = \frac{1}{3} Bh = \frac{1}{3} lwh$ *Cone:* $V = \frac{1}{3} Bh = \frac{1}{3} \pi r^2 h$ *Sphere:* $V = \frac{4}{3}\pi r^3$ *Ellipsoid:* $V = \frac{4}{3}\pi abc$		

	Rectangular	Polar	Parametric
Volume of Revolution	**<u>Disk Method</u>** $V = \int_a^b (area\ of\ circle)\ d(thickness)$ *Rotation about the x-axis:* $V = \int_a^b \pi\, [f(x)]^2 dx$ *Rotation about the y-axis:* $V = \int_c^d \pi x^2 dy$	 	
	<u>Washer Method</u> *Rotation about the x-axis:* $V = \int_a^b \pi\, \{ [f(x)]^2 - [g(x)]^2 \}\, dx$	$V = V_{Outer\ Disk} - V_{Inner\ Disk}$	
	<u>Shell Method</u> $V = \int_a^b (circumference)\ (hight)\ dx$ *Rotation about the y-axis:* $V = \int_a^b 2\pi x\ f(x)\ dx$ *Rotation about the x-axis:* $V = \int_c^d 2\pi y\ g(y)\ dy$